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# ON INVARIANT THEORY UNDER RESTRICTED GROUPS

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A previous paper, 'Invariant theory, tensors and group characters', dealt with invariant theory under the full linear group. In this paper the methods are extended to restricted groups of transformation such as the orthogonal group.

A knowledge of the characters of the group is shown to be an essential preliminary to any adequate study of invariants under the group. The characters of the orthogonal and symplectic groups, previously obtained by Schur and Weyl by transcendental methods involving group integration, are here obtained by methods entirely algebraic.

Concerning transformation groups with a system of fundamental tensors, a fundamental theorem is proved that every concomitant may be obtained by multiplication and contraction of ground-form tensors, tensor variables, fundamental tensors and the alternating tensor.

A characteristic analysis is developed, involving the operation denoted by $\otimes$, which enables the numbers and types of the concomitants of any given degree in any system of ground forms to be predicted. The determination of the actual concomitants is also discussed.

Application is made for the orthogonal group to the quadratic, the ternary cubic and the quaternary quadratic complex; for the ternary symplectic group, to the quadratic, the linear complex and the quadratic complex. Various applications are also made for intransitive and imprimitive groups of transformation.

## 1. Introduction

In a previous paper (Littlewood 1944*c*) the use of tensors in invariant theory was discussed, and an *S*-functional analysis was developed which enabled one to predict the types of concomitants of any given degree in any set of ground forms over the full linear group. Group-substitutional methods of determining the actual concomitants were also described.

It is the purpose of this paper to extend the methods of the previous paper to problems in which the group of transformations is not the full linear group, but some subgroup such as the orthogonal group. In the case when this is the group of transformations which leaves invariant a set of fundamental tensors, it is proved that every concomitant may be obtained by multiplying and contracting ground-form tensors, tensor variables, fundamental tensors and the alternating tensor.

It is shown that no adequate treatment of invariant theory under a restricted group can be attempted without a knowledge of the characters of the group. The characters of the orthogonal and symplectic groups which have previously been obtained by Schur and Weyl by transcendental methods involving group integration, are here obtained by methods which are purely algebraic.

The *S*-functional analysis which was developed in the preceding paper is then adapted to a restricted group and illustrated with reference to the orthogonal and symplectic groups. An application is made to the concomitants of quaternary quadratic complex over the full linear group, the results of which have drawn attention to certain omissions in the complete list of irreducible concomitants obtained by Turnbull (1937).

Intransitive and imprimitive transformation groups are also discussed.

## 2. The nature of the restricted group

The most frequently discussed restricted group of transformations is the *orthogonal group*. The *special orthogonal group* in $n$ variables $x_1, x_2, \ldots, x_n$ is the group of transformations which leaves the quadratic form

$$x_1^2 + x_2^2 + \ldots + x_n^2$$

invariant. The group has been widely discussed and the characters of the group have been obtained by Schur (1924).

The *general orthogonal group* is the group which leaves invariant an arbitrary non-singular quadratic form

$$\Sigma g_{ij} x_i x_j.$$

This may be transformed into the special orthogonal group by a fixed transformation (Littlewood 1940), which will, however, be complex if the signatures of the two forms are not the same. The two groups thus have the same characters, and except for considerations of reality which must be examined in the light of the signature of the form, the examination of the one case will in general give full information concerning the other.

The orthogonal group may be regarded as a special case of a more general class of groups. Suppose that there are $n$ variables, $x_1, \ldots, x_n$, and a set of $r$ forms, $f_1, \ldots, f_r$; each involving any number $\leqslant n$ of sets of variables each cogredient with $x_1, \ldots, x_n$. It may be assumed that these are Clebsch forms (Littlewood 1944*c*). They are called the *fundamental forms*, and their tensors of coefficients are the *fundamental tensors*.

There will be a group of transformations on the variables $x_1, \ldots, x_n$ which leaves each of these $r$ forms absolutely invariant.†

For the orthogonal group there is one such form which is of type $\{2\}$, i.e. it is a quadratic. For the rotation group a second form must be added of type $\{1^n\}$, i.e. the determinant of $n$ sets of cogredient variables.

The case when there is one fundamental form which is a non-singular form of type $\{1^2\}$ has been called by Weyl the *symplectic group*, and the characters of this group have been obtained by Weyl (1939, p. 216).

Returning to the general case, if the forms $f_1, \ldots, f_r$ are absolutely invariant for the group, then clearly, every concomitant of this system of forms will also be left invariant. The converse may be stated as follows.

*If every transformation which leaves $f_1, f_2, \ldots, f_r$ invariant also leaves a form $g$ invariant, then $g$ may be expressed as a concomitant of $f_1, f_2, \ldots, f_r$.*

No proof has yet been found for this in the general case. On the other hand, no exception has been found. It is certainly true for the orthogonal and symplectic groups.

Following the preceding definition of the group in terms of the fundamental forms, the failure to prove this theorem would invalidate the proof of the fundamental theorem in § 3. To avoid this difficulty, the fundamental forms with reference to the group are defined instead, as follows.

*The fundamental forms of a group consist of a minimal system of forms $f_1, f_2, \ldots, f_r$ which are left invariant by the group and such that every form which is left invariant by the group is a concomitant of these forms.*

† See Klein's *Erlanger Program*, **1**, 409.

The above method of defining the group with its fundamental forms is the most convenient if the group of transformations is transitive and primitive.

The transitive condition breaks down for two different types of group which will be called respectively *intransitive* and *semi-transitive* groups.

For an intransitive group, the set of $n$ variables separates into two or more complementary sets, each set being transformed separately by the transformations of the group.

For a semi-transitive group† there is one subset of variables which is transformed within itself, but there is no complementary set with the same property. Examples of this type of group are, first, that generated by transformations of the form

$$\left.\begin{aligned} x' &= ax+by, \\ y' &= \quad\;\; cy; \end{aligned}\right\}$$

secondly, the affine group; and thirdly, the set of transformations in $n$-space from one set of rectangular Cartesian co-ordinates to another.

Representations of a semi-transitive group are usually semi-reducible without being reducible, and consequently the group character does not uniquely define the representation. The methods of this paper are not applicable without modification to semi-transitive groups, and these will not be discussed further.

For imprimitive groups the set of $n$ variables separate into two or more subsets. Transformations of the group either transform the variables separately in each subset, or else permute these subsets.

To obtain the characters of an intransitive group it is more convenient to start, not with the full linear group in all the variables, but with the direct product of the groups of transformation on each transitive set of variables. The characters of a direct product are the products of the characters.

An imprimitive group will have an intransitive subgroup of finite index, the quotient group being a permutation group. The characters of the imprimitive group can usually be obtained without difficulty from those of this intransitive subgroup.

In the main body of this paper the case of a primitive group of transformations will be considered. In § 11 some intransitive and imprimitive groups will be discussed briefly.

*Simple and complex forms and tensors*

Over the full linear group, an algebraic form, or its tensor of coefficients, is said to be *simple* if the matrix of transformation of the coefficients is irreducible. The matrix of transformation of a simple form or a simple tensor is thus an irreducible invariant matrix of the matrix of the basic transformation, i.e. the matrix of transformation of the coefficients in a linear form.

For $n$ variables there is an irreducible invariant matrix corresponding to each partition $(\lambda) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n)$ in $\leqslant n$ parts. The algebraic form may be specified by the spur of the matrix of transformation of its coefficients, which is equal to the $S$-function $\{\lambda\}$ of the latent roots of the matrix of the basic transformation. The form is then said to be of type $\{\lambda\}$, and $\{\lambda\}$ is the character of the full linear group.

† Weyl (1939, p. 48) uses the term 'step transformations'.

A form which is simple over the full linear group may, however, be complex over a restricted group. If so, then the matrix of transformation of the coefficients is reducible, and expressible, by a fixed transformation, as the direct sum of a set of irreducible matrix representations of the group. Since the spur of a direct sum is the sum of the spurs, then

$$\{\lambda\} = \Sigma[\mu],$$

where $[\mu]$ denotes a simple character of the restricted group. The form of type $\{\lambda\}$ is expressible as a sum of forms each of which is simple over the restricted group, in exact correspondence with this equation.

Thus a quaternary quadratic has 10 coefficients, and is a simple form over the full linear group. But over the special orthogonal group the quadratic $\Sigma a_{ij} x_i x_j$ has the linear invariant

$$\tfrac{1}{4}(a_{11}+a_{22}+a_{33}+a_{44})\,(x_1^2+x_2^2+x_3^2+x_4^2).$$

The quadratic will be simple over the orthogonal group only if this linear invariant is removed. The simple orthogonal quaternary quadratic has thus only 9 instead of 10 coefficients.

This corresponds to the equation connecting the characters of the two groups

$$\{2\} = [2]+[0].$$

The term $[0] = 1$ corresponds to the linear invariant, $[2]$ to the simple orthogonal quadratic.

Similarly, corresponding to the equation

$$\{2^2\} = [2^2]+[2]+[0],$$

the full Riemann Christoffel tensor separates into three parts as one separates from it the Einstein tensor $G_{\mu\nu}$, and the invariant $G$.

Clearly no adequate account of invariant theory over a restricted group, not even a knowledge of which forms are simple and which complex, can be obtained without a knowledge of the group characters, or equivalently, of the irreducible representations.

The problem of determining the characters of the group is equivalent to the determination of the linear concomitants of a form of type $\{\lambda\}$.

## 3. The fundamental theorem

Henceforward the tensor notation will be used. In particular variables, being contragredient to the coefficients, will be written with an upper instead of a lower suffix. The summation convention is adopted, namely, that every suffix repeated in an upper and a lower position is summed for all possible values. Since tensors with an arbitrary number of suffixes will be dealt with, it is convenient to represent these by one symbol enclosed in round brackets, e.g.

$$P^{(i)}_{(j)} \equiv P^{i_1 i_2 \ldots i_q}_{j_1 j_2 \ldots j_r}.$$

Further, when a transformation of the variables is introduced, a factor $\xi^{i'}_i$ will be introduced for every upper suffix $i$, and a factor $\eta^j_{j'}$ for every lower suffix $j$. $(\xi^{i'}_i)$ or $(\eta^j_{j'})$ will be used to denote the product of all such factors for every upper or every lower index. Thus, after transformation $P^{(i)}_{(j)}$ becomes $P^{(i)}_{(j)}\,(\xi^{i'}_i)\,(\eta^j_{j'})$.

For the full linear group it was proved in the preceding paper (Littlewood 1944*c*), as the fundamental theorem, that every concomitant of a given set of ground forms may be obtained by multiplying together tensor coefficients, tensor variables, the alternating tensor as required, and contracting. The corresponding theorem for a restricted group is as follows:

Fundamental Theorem. *Every concomitant of a set of ground forms under the group which leaves a set of fundamental tensors invariant, may be obtained by multiplying together tensor coefficients, tensor variables, fundamental tensors and the alternating tensor as required, and contracting.*

Every concomitant may be expressed as a sum of concomitants each of which is simple and homogeneous in the coefficients of each ground form. It is assumed that only one such simple form is considered.

Let the variables undergo the non-singular transformation

$$x'^j = x^i \xi^j_i, \quad x^j = x'^i \eta^j_i.$$

Let the concomitant be expressed in the form

$$f = P^{(i)}_{(j)} a_{(\alpha)} b_{(\beta)} \dots x^{(\gamma)}, \tag{3·1}$$

where $a_{(\alpha)}$, $b_{(\beta)}$, ..., are tensor coefficients and $x^{(\gamma)}$ is a Clebsch variable tensor. The symbol $P^{(i)}_{(j)}$ is a numerical coefficient which is given an upper suffix equal to every lower suffix in the tensor coefficients, and a lower suffix equal to every upper suffix in the Clebsch variable tensor. Summation is assumed to take place between the repeated suffixes.

The suffixes $(\alpha)$, $(\beta)$, $(\gamma)$, etc., satisfy certain symmetrizing relations, corresponding to the types of the various ground forms, the type of the Clebsch tensor, and to the fact that the interchange of coefficients from the same ground form will not alter the expression. Corresponding to any such symmetrizing relation, there will be another symmetrizer which will annihilate this symmetrizer. Correspondingly, a set of coefficients, $P^{(i)}_{(j)}$ may be chosen such that $P^{(i)}_{(j)} a_{(\alpha)} b_{(\beta)} \dots x^{(\gamma)}$ gives zero. The expression (3·1) for the concomitant $f$ is therefore not unique.

To obtain uniqueness, however, one need only operate on the coefficients $P^{(i)}_{(j)}$ with all the symmetrizing relations, satisfied by the suffixes $(\alpha)$, $(\beta)$, $(\gamma)$, etc. If it is assumed that this operation has been effected, then the coefficients $P^{(i)}_{(j)}$ are uniquely determined by the concomitant $f$.

Since $f$ is a concomitant of the ground forms, it is an algebraic form which is either an absolute or a relative invariant. If it is a relative invariant an absolute invariant may be obtained from it by introducing the alternating tensor with the appropriate Clebsch tensor. Hence it may be assumed to be an absolute invariant. Since the coefficients $P^{(i)}_{(j)}$ are uniquely determined by the concomitant, these coefficients are absolutely invariant.

When a transformation of the variables is made, then

$$a'_{(\alpha')} = a_{(\alpha)} (\eta^{\alpha}_{\alpha'}), \quad b'_{(\beta')} = b_{(\beta)} (\eta^{\beta}_{\beta'}), \quad x'^{(\gamma')} = x^{(\gamma)} (\xi^{\gamma'}_{\gamma}).$$

Hence $$P^{(i)}_{(j)} = P^{(i')}_{(j')} (\xi^{\gamma'}_{\gamma}) (\eta^{\alpha}_{\alpha'}) (\eta^{\beta}_{\beta'}) \dots = P^{(i')}_{(j')} (\xi^{j'}_{j}) (\eta^{i}_{i'}),$$

since the upper and lower suffixes of $P$ are respectively equal to the lower and upper suffixes of the tensor component of which it is the coefficient.

Thus $$P^{(i')}_{(j')} = P^{(i)}_{(j)} (\eta^{j}_{j'}) (\xi^{i'}_{i}),$$

so that $P^{(i)}_{(j)}$ is a tensor, which, as has already been seen, is absolutely invariant under the group of transformations. Hence it is a concomitant tensor of the fundamental tensors under the full linear group. By the fundamental theorem for the full linear group it can be obtained from the fundamental tensors and the alternating tensor by multiplication and contraction. By substituting such an expression for it in equation (3·1) the proof of the fundamental theorem for restricted groups is obtained.

## 4. The determination of the characters

The characters of the orthogonal and symplectic groups have been found by Schur (1924) and Weyl (1939) respectively. The method used is transcendental, and depends on integration over the group manifold. These characters, however, may be obtained by purely algebraic methods, and this will be carried out by means of the fundamental theorem of the previous section. This algebraic method would seem to offer a better prospect of successful application to other restricted groups than the method of group integration.

The characters of the orthogonal group in $n$ variables which leaves invariant a non-singular quadratic form $g_{ij}x^i x^j$ are first obtained. Then put $n = 2\nu$ or $n = 2\nu+1$ according as $n$ is even or odd.

Let the characteristic equation of an orthogonal transformation be

$$\zeta^n - a_1\zeta^{n-1} + a_2\zeta^{n-2} - \ldots + (-1)^n a_n = 0.$$

This will be a reciprocal equation, but otherwise, apart from reality conditions which involve inequalities rather than equalities, the coefficients will be unrestricted. Thus $a_1, a_2, \ldots, a_\nu$ will be algebraically independent. It follows that the $S$-functions of the latent roots, corresponding to partitions into $\leqslant \nu$ parts, will be linearly independent. These $S$-functions, which are the characters of the full linear group, are each expressible as a sum of characters of the orthogonal group. Since they are linearly independent, each $S$-function $\{\lambda\}$ will on decomposition yield at least one independent character of the orthogonal group, which will be denoted by $[\lambda]$.

To determine the manner of separation of $\{\lambda\}$ into characters of the orthogonal group the linear concomitants of a tensor $A_{(i)}$ of type $\{\lambda\}$ are found.

These linear concomitants will be obtained by contracting $A_{(i)}$ with the fundamental tensor $g_{ij}$ or its reciprocal $g^{ij}$, or the alternating tensor $E^{(j)} = E^{j_1 j_2 \ldots j_n}$, or all of these taken any number of times.

If a contraction is formed between $A_{(i)}$ and $g^{ij}$ involving one suffix only, the effect will be to replace one cogredient suffix of $A_{(i)}$ by a contragredient suffix. Similarly contraction with $g_{ij}$ may be used to convert a contragredient suffix into a cogredient suffix.

If $A_{(i)}$ is contracted with the alternating tensor, contraction may take place with respect to a set of $r$ suffixes provided that $A_{(i)}$ is alternating with respect to these $r$ suffixes. The effect will be to replace these $r$ cogredient suffixes by $(n-r)$ contragredient suffixes.

Apart from contraction with respect to both suffixes of $g^{ij}$, which case will be considered later, the only procedure by means of which one can return to a cogredient tensor is the conversion of $r$ cogredient suffixes into $(n-r)$ contragredient suffixes by means of the alternating tensor, and the conversion of these $(n-r)$ suffixes into cogredient suffixes again by means of the tensor $g_{ij}$. The resulting tensor will be alternating in these $(n-r)$ suffixes.

The maximum number of suffixes in which the tensor $A_{(i)}$ is alternating is equal to the number of parts in the corresponding partition $\{\lambda\}$ which is $\leqslant \nu$. Hence if $n = 2\nu+1$, or if the number of parts in $\{\lambda\}$ is $<\nu$, $n-r$ must be greater than $\nu$. The resulting tensor being alternating in $(n-r)>\nu$ suffixes will correspond to a partition into $>\nu$ parts. The method will not give a concomitant of the type sought.

Only if $n = 2\nu$ and $\{\lambda\}$ contains $\nu$ parts will the method be available. In this case the set of $\nu$ suffixes are converted into another set of $\nu$ suffixes. A tensor of the same type is obtained. These operations are reversible and exhibit the complete tensor in a different form.

Now consider the case in which contraction takes place with respect to both suffixes of $g^{ij}$. The fundamental tensor may be used any number of times, and these tensors taken together form a tensor concomitant of $g^{ij}$ which will be denoted by $G^{(j)}$. Suppose that this tensor is of type $\{\delta\}$. Suppose also that the contraction is further contracted with a Clebsch tensor $X^{(k)}$ of type $\{\mu\}$ to form an algebraic form of type $[\mu]$. This algebraic form is obtained by the contraction of all the suffixes in

$$A_{(i)}\,G^{(j)}X^{(k)}.$$

If this is possible then the product of the tensors $G^{(j)}X^{(k)}$ has a subtensor of contragredient type $\{\lambda\}$. Thus $\{\lambda\}$ must appear in the product $\{\delta\}\{\mu\}$. If $\Gamma_{\delta\mu\lambda}$ is the coefficient† of $\{\lambda\}$ in $\{\delta\}\{\mu\}$, then $G^{(j)}X^{(k)}$ contains $\Gamma_{\delta\mu\lambda}$ subtensors of type $\{\lambda\}$, and $\Gamma_{\delta\mu\lambda}$ concomitants of type $[\mu]$ may be obtained corresponding to the concomitant $G^{(j)}$ of $g^{ij}$. Summation should then be taken with respect to all concomitants of $g^{ij}$.

The generating function (Littlewood 1940, p. 238) for the concomitants of a quadratic is, if $\{\lambda\}$ denotes an $S$-function of $\alpha_1, \alpha_2, \ldots, \alpha_n$,

$$\frac{1}{\Pi(1-\alpha_i^2)\,\Pi(1-\alpha_i\alpha_j)} = 1+\Sigma\{\delta\},$$

where $\{\delta\}$ is summed for all partitions into even parts only, i.e.

$$1+\{2\}+\{4\}+\{2^2\}+\{6\}+\{42\}+\ldots.$$

The tensor $A_{(i)}$ thus has linear concomitants of each type $[\mu]$ which appears in the series $\Sigma\Gamma_{\delta\mu\lambda}[\mu]$. When these linear concomitants are subtracted from $A_{(i)}$ there remains a tensor of type $[\lambda]$. Hence the following relation between the characters is obtained:

$$\{\lambda\} = [\lambda]+\Sigma\Gamma_{\delta\mu\lambda}[\mu], \tag{4·1}$$

where $\Gamma_{\delta\mu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\delta\}\{\mu\}$, and summation is taken with respect to all partitions $\{\delta\}$ into even parts only.

The equation (4·1) is one of the known formulae for the characters of the orthogonal group. It has previously been deduced‡ from Schur's determinantal form for the characters. A reversal of the reasoning would derive Schur's determinantal form from equation (4·1).

Schur's forms for the characters are not important here. It is sufficient to quote§ the formula expressing the orthogonal group character in terms of $S$-functions, namely,

$$[\lambda] = \{\lambda\}+\Sigma(-1)^{\frac{1}{2}p}\,\Gamma_{\gamma\mu\lambda}\{\mu\}, \tag{4·2}$$

† The symbol $g_{\delta\mu\lambda}$ has previously been used, but it has been changed thus to $\Gamma_{\delta\mu\lambda}$ to avoid confusion with $g_{ij}$.

‡ Littlewood, 1940, p. 240, II. In this theorem $\{\eta\}$ is a misprint for $[\eta]$.

§ Littlewood, 1940, p. 240, I.

where $\{\gamma\}$, a partition of $p$, is summed for all partitions which in Frobenius's nomenclature are of one of the forms

$$\begin{pmatrix} r+1 \\ r \end{pmatrix}, \quad \begin{pmatrix} r+1, & s+1 \\ r, & s \end{pmatrix}, \quad \begin{pmatrix} r+1, & s+1, & t+1 \\ r, & s, & t \end{pmatrix}, \quad \ldots.$$

These partitions $\{\gamma\}$ appear in the expansion

$$\Pi(1-\alpha_i^2)\,\Pi(1-\alpha_i\alpha_j) = 1+\Sigma(-1)^{\frac{1}{2}p}\{\gamma\} = 1-\{2\}+\{31\}-\{41^2\}-\{3^2\}+\{51^3\}+\{431\}-\ldots.$$

For the full orthogonal group the alternating tensor is not an absolute, but a relative invariant. It gives us the character which is $+1$ for a transformation of positive determinant, and $-1$ for a transformation of negative determinant. This character is denoted by $[0]^*$.

For every character $[\lambda]$ of the orthogonal group there is an *associated* character $[\lambda]^*$ defined by

$$[\lambda]^* = [\lambda]\,[0]^*.$$

When $[\lambda]$ corresponds to a partition into $\nu$ non-zero parts however, and $n = 2\nu$, the process of contracting $\nu$ suffixes with the alternating tensor and lowering the replacing $\nu$ contragredient suffixes by means of $g_{ij}$, will exhibit the corresponding tensor $A^{(i)}$ as a tensor of type $[\lambda]^*$, because of the relative invariance of the alternating tensor used. It follows that

$$[\lambda] = [\lambda]^*.$$

Expressed differently, this indicates that for a partition $[\lambda]$ with $\nu$ non-zero parts, $[\lambda] = 0$ for every transformation of negative determinant.

If $\{\lambda\}$ corresponds to a partition into $>\nu$ parts, the corresponding tensor $A_{(i)}$ will have a set of $r>\nu$ suffixes in which it is alternating. These may be converted, as shown above, into a set of $n-r\leqslant\nu$ suffixes. The resulting tensor will be of lower rank. It may not be simple over the full orthogonal group, but if it is separated into simple parts, each may be treated in the same way until all tensors obtained correspond to partitions into $\leqslant\nu$ parts.

This completes the account of the characters of the full orthogonal group. There remains only to describe the behaviour of the characters $[\lambda]$ corresponding to partitions into $\nu$ non-zero parts when $n = 2\nu$ and the group is restricted to proper rotations.

If a set of $\nu$ suffixes of a tensor $A_{(i)}$ of type $\{\lambda\}$, in which the tensor is alternating, is contracted with $\nu$ suffixes of the alternating tensor, and the remaining $\nu$ suffixes are lowered by the contraction with $g_{ij}$, a tensor is obtained also of type $\{\lambda\}$. Dividing by $\nu!$ this may be called tensor $B_{(i)}$. The same operation on the tensor $B_{(i)}$ produces the tensor $(-1)^{\nu}A_{(i)}$.

If $\nu$ is even, put $\qquad A_{(i)} = \tfrac{1}{2}(A_{(i)}+B_{(i)})+\tfrac{1}{2}(A_{(i)}-B_{(i)})$;

if $\nu$ is odd, put $\qquad A_{(i)} = \tfrac{1}{2}(A_{(i)}+iB_{(i)})+\tfrac{1}{2}(A_{(i)}-iB_{(i)})$.

The tensor $A_{(i)}$ is thus separated into two subtensors which remain separate so long as the group is restricted to pure rotations. An orthogonal transformation of negative determinant, however, by changing the sign of the alternating tensor, will interchange these two parts.

This procedure will have the same effect whatever set of $\nu$ suffixes is taken from $A_{(i)}$ provided that $A_{(i)}$ is alternating in this set. This will be seen if one operates simultaneously on two sets of $\nu$ suffixes. This will involve the square of the alternating tensor. But the square of the alternating tensor is a concomitant of the quadratic $g^{ij}$, and can be expressed in terms of $g^{ij}$. Hence the whole operation can be expressed in terms of $A_{(i)}$ and the tensors $g^{ij}$ and $g_{ij}$. In fact the suffixes are first raised by $g^{ij}$ and then lowered by $g_{ij}$. The tensor $A_{(i)}$ is unaltered.

The case of the symplectic group is similar but rather simpler. The invariant linear complex is taken as $h_{ij}x^{ij}$, where $x^{ij}$ is the Clebsch variable tensor of type $\{1^2\}$. If this is to be a non-singular form there must be an even number of variables. Put $n = 2\nu$. The alternating tensor is a concomitant of $h_{ij}$, so that every transformation has determinant $+1$.

As with the orthogonal group, any suffix of a tensor can be raised or lowered by contracting with one of the suffixes of $h_{ij}$ or its reciprocal $h^{ij}$. As with the orthogonal group also, if $A_{(i)}$ is alternating in a set of $r > \nu$ suffixes, these may be contracted with the alternating tensor, and the remaining $n-r$ suffixes lowered using $h_{ij}$. Thus every tensor can be expressed in terms of tensors of types corresponding to partitions into $\leqslant \nu$ parts.

In particular a tensor of type $\{1^r\}$ is equivalent to a tensor of type $\{1^{n-r}\}$. The characteristic equation is thus a reciprocal equation, just as with the orthogonal group.

The contractions with both suffixes of the fundamental tensor are obtained in the same way as for the orthogonal group. The invariant form is

$$A_{(i)}H^{(j)}X^{(k)},$$

where $H^{(j)}$ is a concomitant of $h^{ij}$.

The generating function for the concomitants of a linear complex is

$$\frac{1}{\Pi(1-\alpha_i\alpha_j)} = 1+\Sigma\{\beta\}, \tag{4·3}$$

where $\{\beta\}$ is summed for all partitions in which each part is repeated an even number of times, i.e.

$$1+\{1^2\}+\{2^2\}+\{1^4\}+\{3^2\}+\{2^2\,1^2\}+\dots.$$

The corresponding formula to (4·1), if $\langle\mu\rangle$ denotes a character of the symplectic group, is thus

$$\{\lambda\} = \langle\lambda\rangle + \Sigma\Gamma_{\beta\mu\lambda}\langle\mu\rangle, \tag{4·4}$$

where $\Gamma_{\beta\mu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\beta\}\{\mu\}$, and $\{\beta\}$ is summed for the $S$-functions in the series (4·3).

If $A$ is the matrix of the basic transformation, let

$$f(x) = |\,I-xA\,| = 1-a_1x+a_2x^2-a_3x^3+\dots+a_nx^n.$$

Let $x_1, x_2, \dots, x_\nu$ be a set of $\nu$ variables, and let $\{\lambda; x\}$ denote the $S$-function $\{\lambda\}$ of the quantities $x_1, x_2, \dots, x_\nu$. Let $f_r$ denote $f(x_r)$. Then (Littlewood 1940, p. 240)

$$\frac{1}{f_1f_2\dots f_r} = 1+\Sigma\{\lambda\}\{\lambda; x\}.$$

Expanding $\Pi(1-x_rx_s)/f_1f_2\dots f_\nu$ in terms of the $S$-functions of the $x_r$'s, let

$$\Pi(1-x_rx_s)/f_1f_2\dots f_\nu = 1+\Sigma F^{(\lambda)}\{\lambda; x\},$$

the coefficients $F^{(\lambda)}$ being symmetric functions of $\alpha_1, \alpha_2, \dots, \alpha_n$, the latent roots of $A$.

Then
$$\frac{1}{f_1f_2\dots f_\nu} = 1+\Sigma\{\lambda\}\{\lambda; x\} = \frac{1}{\Pi(1-x_rx_s)}\frac{\Pi(1-x_rx_s)}{f_1f_2\dots f_\nu}$$
$$= [1+\Sigma\{\beta; x\}]\,[1+\Sigma F^{(\lambda)}\{\lambda; x\}].$$

The coefficient of $\{\lambda; x\}$ in this expansion is

$$\{\lambda\} = F^{(\lambda)} + \Sigma \Gamma_{\beta\mu\lambda} F^{(\mu)}.$$

Comparing this equation with (4·4) we see that

$$\langle\lambda\rangle = F^{(\lambda)}.$$

Hence we have the following generating function for the characters of the symplectic group

$$\frac{\Pi(1-x_r x_s)}{f_1 f_2 \dots f_\nu} = 1 + \Sigma\langle\lambda\rangle\{\lambda; x\}. \tag{4·5}$$

Using $\Delta(x_1, \dots, x_\nu)$ to denote $\prod_{r<s}(x_r - x_s)$, (4·6) is obtained on multiplying (4·5) by this,

$$\frac{\Pi(1-x_r x_s)\,\Delta(x_1, \dots, x_\nu)}{f_1 f_2 \dots f_\nu} = \Sigma \pm \langle\lambda\rangle\, x_1^{\lambda_1+\nu-1} x_2^{\lambda_2+\nu-2} \dots x_\nu^{\lambda_\nu}, \tag{4·6}$$

the sum including all partitions $(\lambda)$ and all permutations of the lower suffixes with the negative sign for a negative permutation.

Weyl's determinantal form (Weyl 1939, p. 219) for $\langle\lambda\rangle$ may readily be deduced from (4·6).

The symplectic group characters may be expressed in terms of the $S$-functions $\{\lambda\}$ as follows:

$$1 + \Sigma\langle\lambda\rangle\{\lambda; x\} = \Pi(1-x_r x_s)\frac{1}{f_1 f_2 \dots f_\nu} = [1 + \Sigma(-1)^{\frac{1}{2}p}\{\alpha; x\}]\,[1 + \Sigma\{\lambda\}\{\lambda; x\}],$$

where $\{\alpha\}$, a partition of $p$, is summed for the partitions which in Frobenius's nomenclature are of the forms

$$\begin{pmatrix} r \\ r+1 \end{pmatrix}, \quad \begin{pmatrix} r, & s \\ r+1, & s+1 \end{pmatrix}, \quad \begin{pmatrix} r, & s, & t \\ r+1, & s+1, & t+1 \end{pmatrix}, \quad \dots,$$

i.e. the partitions in the series

$$\Pi(1-x_r x_s) = 1 - \{1^2\} + \{21^2\} + \{2^3\} - \{32^21\} - \dots.$$

Comparing coefficients of $\{\lambda; x\}$, then

$$\langle\lambda\rangle = \{\lambda\} + \Sigma(-1)^{\frac{1}{2}p}\,\Gamma_{\mu\alpha\lambda}\{\mu\}. \tag{4·7}$$

Since the determinant of a symplectic transformation is $+1$, the difficulties which arise from distinguishing between the orthogonal and the rotation groups do not arise with the symplectic group. Also the character $\langle\lambda\rangle$ when the partition $(\lambda)$ contains $\nu$ non-zero parts, does not separate into two conjugate characters as with the rotation group, as the following consideration will indicate. Let $A_{(i)}$ be a tensor of type $\langle\lambda\rangle$, and suppose that a set of $\nu$ suffixes in which $A_{(i)}$ is alternating is contracted with the alternating tensor, and the remaining $\nu$ suffixes from the alternating tensor are lowered by contraction with $h_{ij}$. Since the alternating tensor is a concomitant of $h^{ij}$, the whole operation may be expressed in terms of $h^{ij}$ and $h_{ij}$. The effect then is to raise the $\nu$ suffixes by contracting with $h^{ij}$, and then to lower them again by contracting with $h_{ij}$. This will leave the tensor $A_{(i)}$ unaltered.

## 5. The concomitants of forms or tensors under a restricted group

Let $\zeta$ be a character of the restricted group and let $A_{(i)}$ be a tensor of type $\zeta$. The matrix of transformation of $A_{(i)}$, which will be denoted by $T(\zeta)$, will have spur equal to $\zeta$. The product of $r$ tensors each equal to $A_{(i)}$ has for its matrix of transformation the $r$th induced matrix of $T(\zeta)$. Using the symbol $\otimes$ as in the 'new multiplication of $S$-functions' (Littlewood 1936, 1940, p. 206), the spur of this $r$th induced matrix is denoted by $\zeta\otimes\{r\}$. Let this be expressed as a sum of simple characters of the restricted group in the form

$$\zeta\otimes\{r\} = \Sigma\eta.$$

Then clearly there is a concomitant of $A_{(i)}$ of degree $r$ and of type $\eta$ corresponding to each term $\eta$ in the summation.

Just as in the case of the full linear group, if $A_{(i)}$ and $B_{(i)}$ are two ground-form tensors of type $\zeta$ and $\xi$ respectively, then the types of concomitant of degree $r$ in $A_{(i)}$ and $s$ in $B_{(i)}$ will correspond to the terms $\eta$ in the expansion

$$[\zeta\otimes\{r\}]\,[\xi\otimes\{s\}] = \Sigma\eta,$$

and similarly for three or more ground forms.

To determine the types of concomitant under the restricted group, then, one must be able to express the product of two characters as a sum of simple characters, and also to evaluate $\zeta\otimes\{r\}$ as a sum of simple characters.

The first may usually be accomplished when the characters are known. To evaluate $\zeta\otimes\{r\}$ three methods are available.

The first method is from first principles. $\zeta$ is expressed as a sum of powers and products of the latent roots of the matrix of the basic transformation. These are the latent roots of $T(\zeta)$. The sum of the powers and products of these of degree $r$ is formed and then expressed as a sum of characters. This method has been used to find the concomitants of a basic spin tensor under the orthogonal group (Littlewood 1944*b*).

Secondly, if the characters $\zeta$ are expressible linearly in terms of $S$-functions, the known technique for $S$-functions may be used to evaluate $\zeta\otimes\{r\}$. This method will be illustrated with reference to the orthogonal and symplectic groups.

Thirdly, a technique may be developed for the special group in the same manner as a technique was developed for $S$-functions for the full linear group. This procedure, however, might be very difficult.

When possible, the second method is greatly to be preferred.

When the types of the concomitants are known, the determination of the actual concomitants is made much easier. In the case when the characters are linearly expressible in terms of $S$-functions, as with the orthogonal and symplectic groups, the concomitants of a form of type $\{\lambda\}$ may be obtained by first finding the concomitants under the full linear group, and then separating each concomitant tensor into its simple subtensors. The form in general, however, will be of type $[\lambda]$, where this denotes a character of the restricted group, which implies that all contractions with respect to the fundamental tensors are zero. This will automatically annihilate many of the concomitants that were obtained for the form of type $\{\lambda\}$. Those concomitants which remain will usually be the concomitants of the form of type $[\lambda]$, but this may be checked by comparison with the expansion of $[\lambda]\otimes\{r\}$.

The method will be illustrated.

## 6. The orthogonal and symplectic groups

### (i) *Reducibility*

With the full linear group reducibility is defined with reference to Clebsch forms. The product of two forms of types $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\{\mu_1, \mu_2, \ldots, \mu_n\}$ is then a simple form of type $\{\lambda_1+\mu_1, \lambda_2+\mu_2, \ldots, \lambda_n+\mu_n\}$. For tensors, the product of two tensors of types $\{\lambda\}$ and $\{\mu\}$ respectively is a compound tensor which includes a tensor of type $\{\lambda_1+\mu_1, \ldots, \lambda_n+\mu_n\}$. This last tensor is called the *principal part* of the product of the two tensors.

To bring the invariant theory of tensors in conformity with the invariant theory of algebraic forms, the convention is made that a concomitant is only regarded as *reducible* if it is the principal part of the product of two tensors, or is a linear combination of such principal parts.

For the orthogonal and symplectic groups the product of two simple forms is in general complex. To simplify the concept of reducibility we adopt the same convention as for tensors.

The product of two tensors, or algebraic forms of type $[\lambda_1, \ldots, \lambda_\nu]$ and $[\mu_1, \ldots, \mu_\nu]$ respectively, always includes a tensor or algebraic form of type $[\lambda_1+\mu_1, \ldots, \lambda_\nu+\mu_\nu]$, which is called the *principal part* of the product. The same is true of the symplectic group. A concomitant is *reducible* only if it is the principal part of the product of two concomitants, or is a linear combination of such principal parts.

### (ii) *S-functions with more than ν parts*

By formulae (4·1) and (4·7) any orthogonal group or symplectic group character can be expressed in terms of $S$-functions. By formulae (4·1) and (4·4) $S$-functions can be expressed in terms of orthogonal group or symplectic group characters, but only if the $S$-functions contain $\leqslant \nu$ parts. In the calculation of concomitants there arise also $S$-functions with $>\nu$ parts, for which these formulae are not applicable. The following method may then be used to convert these $S$-functions into $S$-functions with $\leqslant \nu$ parts, and the formulae (4·1) and (4·4) are then applicable. The work involved may sometimes be a trifle laborious as compared with the simplicity of the final formulae, and it is hoped that a more direct method may subsequently be discovered of converting $S$-functions with $>\nu$ parts directly in terms of the orthogonal or symplectic group characters.

The method is equivalent to the conversion of some of the suffixes of a cogredient tensor into contragredient suffixes, to form a mixed tensor, by means of the alternating tensor, and the subsequent lowering of the suffixes again by the use of the fundamental tensor.

The $S$-function is expressed in the form

$$\{r+\lambda_1, r+\lambda_2, \ldots, r+\lambda_\nu, r-\mu_\nu, r-\mu_{\nu-1}, \ldots, r-\mu_1\} \tag{6·1}$$

if $n = 2\nu$, or if $n = 2\nu+1$, in the form

$$\{r+\lambda_1, r+\lambda_2, \ldots, r+\lambda_\nu, r, r-\mu_\nu, r-\mu_{\nu-1}, \ldots, r-\mu_1\}. \tag{6·1$'$}$$

If one ignores a change of sign for a transformation of negative determinant, for the orthogonal group, the value of this $S$-function is independent of $r$. It will be denoted by $\{\lambda; \mu\}$.

Then if $(\delta)$ is any partition of any number $d$, and $(\epsilon)$ is the conjugate partition, the formula is

$$\{\lambda;\mu\} = \{\lambda\}\{\mu\} + \Sigma(-1)^d \Gamma_{\alpha\epsilon\lambda}\Gamma_{\beta\delta\mu}\{\alpha\}\{\beta\}. \qquad (6\cdot2)$$

The summation is with respect to all partitions $(\delta)$ of all numbers $d$ such that $\{\lambda\}$ appears in the product $\{\alpha\}\{\epsilon\}$ with coefficient $\Gamma_{\alpha\epsilon\lambda}$, and $\{\mu\}$ in the product $\{\beta\}\{\delta\}$ with coefficient $\Gamma_{\beta\delta\mu}$.

The proof is a little intricate in comparison with the significance of the theorem. It is as follows:

Taking the conjugates of the partitions $(\lambda)$, $(\mu)$, put

$$(\tilde{\lambda}) = (\phi) \equiv (\phi_1, \ldots, \phi_p), \quad (\tilde{\mu}) = (\psi) \equiv (\psi_1, \ldots, \psi_q).$$

The conjugate of the $S$-function $\{\lambda;\mu\}$ may then be expressed as

$$\{\widetilde{\lambda;\mu}\} = \{n-\psi_q, n-\psi_{q-1}, \ldots, n-\psi_1, \phi_1, \ldots, \phi_p\}.$$

Hence the formulae (Littlewood 1940, p. 89) expressing an $S$-function in terms of the quantities $a_r$ gives

$$\{\lambda;\mu\} = \begin{vmatrix} a_{n-\psi_{q+1-s}-s+t} \\ \cdots\cdots\cdots\cdots\cdots \\ a_{\phi_{s'}-q-s'+t} \end{vmatrix}.$$

The Laplace expansion of this determinant according to the given separation of the rows, gives

$$\{\lambda;\mu\} = \Sigma \pm |a_{n-\psi_{q+1-s}-s+u_t}|\,|a_{\phi_s-q-s+v_t}|,$$

in which for the sequence $u_t$ is taken every combination of $q$ values between 1 and $p+q$, $v_t$ taking the remaining $p$ values.

Now put $$v_t = q+t-\delta_t,$$
then $$u_t = t+\epsilon_{q+1-t},$$

where $(\epsilon_1, \epsilon_2, \ldots, \epsilon_q)$ is the partition conjugate to $(\delta_1, \delta_2, \ldots, \delta_p)$. If $(\epsilon)$ and $(\delta)$ are partitions of $d$, the appropriate sign is $(-1)^d$.

Reversing the order of the rows and columns in the first determinant, and remembering that since the characteristic equation is reciprocal $a_{n-r} = a_r$, then

$$\{\lambda;\mu\} = \Sigma(-1)^d |a_{n-\psi_s+s-t+\epsilon_t}|\,|a_{\phi_s-s+t-\delta_t}| = \Sigma(-1)^d |a_{\psi_s-s+t-\epsilon_t}|\,|a_{\phi_s-s+t-\delta_t}|.$$

These are isobaric determinants (Littlewood 1940, p. 110) and may be expressed in terms of $S$-function by known formulae, which yield the equation

$$\{\lambda;\mu\} = \Sigma(-1)^d \Gamma_{\beta\delta\mu}\Gamma_{\alpha\epsilon\lambda}\{\alpha\}\{\beta\}.$$

By way of example the $S$-function $\{4322\}$ for the orthogonal or symplectic group in six variables is expressed in terms of $S$-functions with not more than three parts. Then†

$$\begin{aligned}\{4,3,2,2\} &= \{2,1,0,0,-2,-2\} \\ &= \{21;2^2\} \\ &= \{21\}\{2^2\} - (\{2\}+\{1^2\})\{21\} + \{1\}\{1^2\} + \{1\}\{2\} - \{1\} \\ &= \{43\}+\{421\}+\{331\}+\{322\}+\{3211\}+\{2^31\}-\{41\}-\{32\}-\{31^2\} \\ &\quad -\{2^21\}-\{32\}-\{31^2\}-\{2^21\}-\{21^3\}+\{21\}+\{1^3\}+\{3\}+\{21\}-\{1\}.\end{aligned}$$

† The convention is adopted, writing $S$-functions with negative parts, as described by Littlewood (1944 *a*).

Now $\{3211\} = \{21;1^2\} = \{32\}+\{31^2\}+\{2^21\}+\{21^3\}-\{3\}-2\{21\}-\{1^3\}+\{1\}.$

Hence

$$\{4322\} = \{43\}+\{421\}+\{3^21\}+\{32^2\}-\{41\}-\{32\}-\{31^2\}+2\{3\}+4\{21\}+2\{1^3\}.$$

## 7. The orthogonal quadratic

The concomitants, up to degree 5 in the coefficients, of a quadratic under the orthogonal group, in 3–6 variables will now be determined.

Under the full linear group, if the concomitants in $n$ variables are known, a knowledge of the concomitants in $r<n$ variables follows. Those concomitants in $n$ variables which correspond to a partition with $>r$ parts are identically zero. The remaining concomitants in $n$ variables give the concomitants in $r$ variables.

With the orthogonal group (and similarly with the symplectic), this is not the case, and the concomitants must be worked out separately in each number of variables.

It will be assumed that the quadratic is simple, i.e. its linear invariant is zero, and thus it is of type [2]. The concomitants of degree $n$ can be determined as follows. The formula (Littlewood 1944*a*)

$$(A-B)\otimes\{\lambda\} = A\otimes\{\lambda\}+\Sigma(-1)^b\,\Gamma_{\alpha\beta\lambda}(A\otimes\{\alpha\})\,(B\otimes\{\tilde{\beta}\})$$

is used, where $(\beta)$ is a partition of $b$, $(\tilde{\beta})$ is the conjugate partition, and $\Gamma_{\alpha\beta\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\alpha\}\{\beta\}$.

Since $\{0\}\otimes\{n\} = \{0\}$, and $\{0\}\otimes\{\lambda\} = 0$

when $\{\lambda\}$ is a partition with more than one part,

$$[2]\otimes\{n\} = (\{2\}-\{0\})\otimes\{n\} = \{2\}\otimes\{n\}-\{2\}\otimes\{n-1\}.$$

The expansion (Littlewood 1944*c*) of $\{2\}\otimes\{n\}$ contains an $S$-function for every partition of $2n$ into even parts only. Hence

$[2]\otimes\{2\} = \{4\}+\{2^2\}-\{2\},$

$[2]\otimes\{3\} = \{6\}+\{42\}+\{2^3\}-\{4\}-\{2^2\},$

$[2]\otimes\{4\} = \{8\}+\{62\}+\{4^2\}+\{42^2\}+\{2^4\}-\{6\}-\{42\}-\{2^3\},$

$[2]\otimes\{5\} = \{10\}+\{82\}+\{64\}+\{62^2\}+\{4^22\}+\{42^3\}+\{2^5\}-\{8\}-\{62\}-\{4^2\}-\{42^2\}-\{2^4\}.$

The expression of these $S$-functions in terms of orthogonal group characters will depend on the number of variables.

For any number of variables $\{4\}-\{2\} = [4].$

For three variables $\{2^2\} = \{2\} = [2]+[0],$

but for more than three variables

$$\{2^2\} = [2^2]+[2]+[0].$$

The following equations are also independent of the number of variables:

$$\{6\}-\{4\} = [6],\quad \{8\}-\{6\} = [8],\quad \{10\}-\{8\} = [10].$$

For three variables, using (6·2),

$$\{n+1, 1\} = \{n; 1\} = \{n+1\}+\{n, 1\}-\{n-1\}$$
$$= [n+1]+\{n, 1\}$$
$$= [n+1]+[n]+[n-1]+\ldots+[2]+[1].$$
$$\{n+2, 2\} = \{n; 2\} = \{n+2\}+\{n+1.1\}+\{n, 2\}-\{n\}-\{n-1, 1\}$$
$$= [n+2]+[n+1]+[n]+\{n, 2\}.$$

Hence
$$\{42\} = [4]+[3]+2[2]+[0],$$
$$\{2^3\} = [0],$$
$$\{2^2\} = \{2\} = [2]+[0],$$
$$\{62\} = [6]+[5]+2[4]+[3]+2[2]+[0],$$
$$\{4^2\} = \{4\} = [4]+[2]+[0],$$
$$\{42^2\} = \{2\} = [2]+[0],$$
$$\{82\} = [8]+[7]+2[6]+[5]+2[4]+[3]+2[2]+[0],$$
$$\{64\} = \{62\},$$
$$\{62^2\} = \{4\},$$
$$\{4^22\} = \{2\}.$$

In four or more variables by (4·1)

$$\{42\} = [42]+[4]+[31]+[2^2]+2[2]+[0],$$
$$\{62\} = [62]+[6]+[51]+[42]+2[4]+[31]+[2^2]+2[2]+[0],$$
$$\{4^2\} = [4^2]+[42]+[4]+[2^2]+[2]+[0],$$
$$\{82\} = [82]+[8]+[71]+[62]+2[6]+[51]+[42]+2[4]+[31]+[2^2]+2[2]+[0],$$
$$\{64\} = [64]+[62]+[53]+[4^2]+[6]+[51]+2[42]+2[4]+[31]+[2^2]+2[2]+[0].$$

In four variables only

$$\{2^3\} = \{2\},$$
$$\{42^2\} = \{2; 2\}$$
$$= \{4\}+\{31\}+\{2^2\}-\{2\}-\{1^2\}$$
$$= [4]+[31]+[2^2]+2[2]+[0],$$
$$\{2^4\} = [0],$$
$$\{62^2\} = \{4; 2\}$$
$$= \{6\}+\{51\}+\{42\}-\{31\}-\{4\}$$
$$= [6]+[51]+[42]+2[4]+[31]+[2^2]+2[2]+[0],$$
$$\{442\} = \{42\}.$$

In five variables the following cases need special calculation:

$$\{422\} = \{2;2^2\}$$
$$= \{42\}+\{321\}+\{2^3\}-\{31\}-\{2^2\}-\{21^2\}+\{1^2\},$$
$$\{321\} = \{21;1^2\}$$
$$= \{32\}+\{31^2\}+\{2^21\}+\{21^3\}-\{3\}-2\{21\}-\{1^3\}+\{1\},$$
$$\{31^2\} = \{2;1^2\} = \{31\}+\{21^2\}-\{2\}-\{1^2\}+\{0\},$$
$$\{21^2\} = \{1;1^2\} = \{21\}+\{1^3\}-\{1\},$$
$$\{21^3\} = \{1;1\} = \{2\}+\{1^2\}-\{0\},$$
$$\{2^21\} = \{1^2;1^2\} = \{2^2\}+\{21^2\}+\{1^4\}-\{2\}-\{1^2\}.$$

Hence $$\{42^2\} = [42]+[4]+[32]+[31]+2[2^2]+2[2]+[0].$$

Similarly $$\{4^22\} = [4^2]+[43]+2[42]+[4]+[32]+[31]+2[2^2]+2[2]+[0],$$

and $$\{622\}-\{422\} = [62]+[52]+[6]+[51]+[42]+[4].$$

For six variables the only case of difficulty is

$$\{42^3\} = \{2;2^2\}$$
$$= \{42\}+\{321\}+\{2^3\}-\{31\}-\{2^2\}-\{21^2\}+\{1^2\}$$
$$= [42]+[4]+[321]+[31]+[2^3]+[2^2]+2[2]+[0].$$

The following formulae may thus be obtained, which indicate the types of concomitant up to degree 5 of a quadratic in three to six variables:

*Three Variables*:

$$[2]\otimes\{2\} = [4]+[2]+[0],$$
$$[2]\otimes\{3\} = [6]+[4]+[3]+[2]+[0],$$
$$[2]\otimes\{4\} = [8]+[6]+[5]+2[4]+2[2]+[0],$$
$$[2]\otimes\{5\} = [10]+[8]+[7]+2[6]+[5]+2[4]+[3]+2[2]+[0].$$

*Four Variables*:

$$[2]\otimes\{2\} = [4]+[2^2]+[2]+[0],$$
$$[2]\otimes\{3\} = [6]+[42]+[4]+[31]+2[2]+[0],$$
$$[2]\otimes\{4\} = [8]+[62]+[6]+[51]+[4^2]+[42]+2[4]+[31]+2[2^2]+2[2]+2[0],$$
$$[2]\otimes\{5\} = [10]+[82]+[8]+[71]+[64]+[62]+3[6]+[53]+2[51]+3[42]+3[4]$$
$$+2[31]+[2^2]+4[2]+[0].$$

*Five Variables*:

$$[2]\otimes\{2\} = [4]+[2^2]+[2]+[0],$$
$$[2]\otimes\{3\} = [6]+[42]+[4]+[31]+[2^2]+2[2]+[0],$$
$$[2]\otimes\{4\} = [8]+[62]+[6]+[51]+[4^2]+2[42]+3[4]+[32]+[31]+2[2^2]+3[2]+2[0],$$
$$[2]\otimes\{5\} = [10]+[82]+[8]+[71]+[64]+2[62]+3[6]+[53]+[52]+2[51]+[4^2]$$
$$+[43]+4[42]+4[4]+[32]+3[31]+3[2^2]+4[2]+2[0].$$

*Six Variables*:

$$[2]\otimes\{2\} = [4]+[2^2]+[2]+[0],$$

$$[2]\otimes\{3\} = [6]+[42]+[4]+[31]+[2^3]+[2^2]+2[2]+[0],$$

$$[2]\otimes\{4\} = [8]+[62]+[6]+[51]+[4^2]+[42^2]+2[42]+3[4]+[321]+[31]+3[2^2]+3[2]+2[0],$$

$$[2]\otimes\{5\} = [10]+[82]+[64]+[62^2]+[4^2 2]+[8]+[71]+2[62]+[53]+[521]+[4^2]+[431]+[42^2]+3[6]+2[51]+5[42]+2[321]+2[2^3]+4[4]+3[31]+3[2^2]+5[2]+2[0].$$

The correctness of these expansions may be checked by the counting of terms in each concomitant. The number of terms in each concomitant is obtained by evaluating the corresponding $[\lambda]$ for the identical transformation (Littlewood 1942). A formula has been given by Schur (Schur 1924; or see Littlewood 1940). It is

$$[\lambda] = 2^{\nu}\Pi[(\lambda_r-\lambda_s-r+s)\,(\lambda_r+\lambda_s+n-r-s)]/[(n-2)!\,(n-4)!\dots 2!],$$

if $n=2\nu$ and $\lambda_\nu \neq 0$, but half this value if $\lambda_\nu = 0$. If $n = 2\nu+1$, then

$$[\lambda] = \Pi[(\lambda_r-\lambda_s-r+s)\,(\lambda_r+\lambda_s+n-r-s)]\,\Pi(2\lambda_r+n-2r)/[(n-2)!\,(n-4)!\dots 3!].$$

Thus for six variables $\qquad [2] = 20.$

Hence $[2]\otimes\{2\}$, $[2]\otimes\{3\}$, $[2]\otimes\{4\}$, $[2]\otimes\{5\}$ are respectively equal to

$$\frac{20.21}{1.2} = 210,\quad \frac{20.21.22}{1.2.3} = 1540,\quad \frac{20.21.22.23}{1.2.3.4} = 8855,\quad \frac{20.21.22.23.24}{1.2.3.4.5} = 42{,}504,$$

for the identical transformation.

Checking $[2]\otimes\{2\}$

$$[4] = [4,0,0] = 4.5.6.1.7.6.1/4!\,2! = 105,$$
$$[2^2] = [2,2,0] = 4.4.3.1.7.4.3/4!\,2! = 84,$$
$$[2] = 20,$$
$$[0] = 1,$$

and $\qquad 210 = 105+84+20+1.$

All the expansions have been checked in this way.

The expansions give the numbers and types of the complete set of concomitants of each degree. The *irreducible* concomitants are obtained by deleting from each list a term corresponding to each reducible concomitant that can be formed by combining irreducible concomitants of lower degree. Thus the following irreducible concomitants are obtained.

*Three Variables*:

Degree 2; [2], [0].
Degree 3; [3], [0].

This is the complete irreducible system for all degrees, and is already known.†

† Cf. Grace & Young (1903, p. 288). The failure of irreducibles at degree $p$ implies none for higher degrees. Grance & Young give a proof in the ternary case (pp. 294 et seq.) which would appear to apply generally.

*Four Variables*:

Degree 2; $[2^2]$, [2], [0].
Degree 3; [31], [2], [0].
Degree 4; [31], $[2^2]$, [0].
Degree 5; [31].

*Five Variables*:

Degree 2; $[2^2]$, [2], [0].
Degree 3; [31], $[2^2]$, [2], [0].
Degree 4; [32], [31], $[2^2]$, $[2^2]$, [2], [0].
Degree 5; [43], [32], [31], [31], $[2^2]$, [0].

*Six Variables*:

Degree 2; $[2^2]$, [2], [0].
Degree 3; [31], $[2^3]$, $[2^2]$, [2], [0].
Degree 4; [321], [31], $[2^2]$, $[2^2]$, [2], [0].
Degree 5; [431], [321], [321], $[2^3]$, [31], [31], $[2^2]$, [2], [0].

In the case of four variables we proceed to augment the above results by proceeding as far as degree 7 to obtain the irreducible system for all degrees:

$$[2]\otimes\{6\} = [12]+[10.2]+[10]+[91]+[84]+[82]+3[8]+[73]+2[71]+[66]+[64] +4[62]+4[6]+[53]+3[51]+2[44]+3[42]+6[4]+[33]+2[31]+3[2^2] +4[2]+3[0].$$

$$[2]\otimes\{7\} = [14]+[12.2]+[12]+[11.1]+[10.4]+[10.2]+3[10]+[93]+2[91]+[86] +[84]+4[82]+4[8]+[75]+2[73]+4[71]+3[64]+4[62]+8[6]+3[53] +5[51]+[44]+6[42]+6[4]+4[31]+2[2^2]+6[2]+2[0].$$

The irreducible concomitants are of the following types:

Degree 6; [4], [33].
Degree 7; [42].

The expansion of $[2]\otimes\{8\}$ yields no irreducible concomitants, and the list for the orthogonal quaternary quadratic is now complete for all degrees.

The explicit concomitants are not difficult to obtain. The concomitants under the full linear group are first obtained. If any of these correspond to a partition into $>\nu$ parts the corresponding tensor is expressed in terms of tensors corresponding to partitions with $\leqslant\nu$ parts by the method described in § 4, in accordance with the formula (6·2). All possible contractions with $g^{ij}$ are then obtained, except that the contraction $A_{ij}g^{ij}$ is avoided which would of course give zero, $A_{ij}$ being the tensor of coefficients of the ground-form quadratic.

The complete set of irreducible concomitants for all cases are too numerous to give here, but one writes down by way of illustration the irreducible concomitants up to degree 5 of an orthogonal quadratic in four variables.

The symbolic notation is used, representing the ground-form quadratic as

$$\alpha_x^2 = \beta_x^2 = \gamma_x^2 = \delta_x^2 = \epsilon_x^2.$$

The symbol $[\alpha\beta]$ is used to denote the contraction $A_{i.} A_{j.} g^{ij}$, the symbol $(\alpha\beta\gamma g)$ to denote

$$E^{ijkp} A_{i.} A_{j.} A_{k.} g_{p.},$$

and $(\alpha\beta\gamma x)$ to denote $$E^{ijkp} A_{i.} A_{j.} A_{k.} g_{pq} x^q.$$

The concomitants may then be expressed as follows:

$$(\alpha\beta \mid xy)^2, \quad [\alpha\beta]\alpha_x\beta_x, \quad [\alpha\beta]^2.$$

$$(\alpha\beta \mid xy)\,\alpha_x\gamma_x[\beta\gamma], \quad (\alpha\beta\gamma x)^2, \quad [\alpha\beta]\,[\beta\gamma]\,[\alpha\gamma].$$

$$(\alpha\beta\gamma g)\,(\alpha\beta\gamma g')\,(g\delta \mid xy)\,g'_x\delta_x, \quad (\alpha\beta\gamma g)\,(\alpha\beta\gamma g')\,(g\delta \mid xy)\,(g'\delta \mid xy), \quad (\alpha\beta\gamma\delta)^2,$$

$$(\alpha\beta \mid xy)\,\alpha_x\gamma_x[\beta\delta]\,[\gamma\epsilon]\,[\delta\epsilon].$$

## 8. The orthogonal ternary cubic

Next the concomitants of a simple ternary cubic are obtained under the orthogonal group up to degree 5 in the coefficients. The cubic, being simple, is of type [3].

Now $$[3] = \{3\} - \{1\},$$

and thus
$$\begin{aligned}[3] \otimes \{2\} &= (\{3\} - \{1\}) \otimes \{2\} \\ &= \{3\} \otimes \{2\} - \{3\}\{1\} + \{1^2\} \\ &= \{6\} + \{42\} - \{4\} - \{31\} + \{1^2\} \\ &= [6] + [4] + [2] + [0].\end{aligned}$$

Similarly

$$\begin{aligned}[3] \otimes \{3\} &= \{3\} \otimes \{3\} - \{3\} \otimes \{2\}\{1\} + \{3\}\{1^2\} - \{1^3\} \\ &= \{9\} + \{72\} + \{63\} + \{52^2\} + \{4^21\} - \{61\} - \{52\} - \{43\} - \{421\} + \{4\} + \{31\} - \{0\}.\end{aligned}$$

Of these $S$-functions
$$\begin{aligned}\{72\} &= [7] + [6] + 2[5] + [4] + [3] + \{32\} \\ &= [7] + [6] + 2[5] + [4] + 2[3] + [2] + [1].\end{aligned}$$

Also

$$\begin{aligned}\{n+3\,.\,3\} &= \{n;\,3\} = \{n\}\{3\} - \{n-1\}\{2\} \\ &= \{n+3\} + \{n+2\,.\,1\} + \{n+1\,.\,2\} + \{n\,.\,3\} - \{n+1\} - \{n\,.\,1\} - \{n-1\,.\,2\} \\ &= [n+3] + [n+2] + 2[n+1] + [n] + [n-1] + \{n\,.\,3\}.\end{aligned}$$

Thus
$$\begin{aligned}\{63\} &= [6] + [5] + 2[4] + [3] + [2] + \{3^2\} \\ &= [6] + [5] + 2[4] + 2[3] + [2] + [1].\end{aligned}$$

The other $S$-functions have been expressed in terms of the orthogonal characters in § 7.

Hence $$[3] \otimes \{3\} = [9] + [7] + [6] + [5] + [4] + 2[3] + [1].$$

Similarly

$$\begin{aligned}[3]\otimes\{4\} &= \{3\}\otimes\{4\}-\{3\}\otimes\{3\}\{1\}+\{3\}\otimes\{2\}\{1^2\}-\{3\}\{1^3\}\\ &= \{12\}+\{10.2\}+\{93\}+\{84\}+\{82^2\}+\{741\}+\{732\}+\{6^2\}+\{642\}+\{4^3\}-\{10\}\\ &\quad -\{91\}-\{82\}-\{73\}-\{721\}-\{73\}-\{64\}-\{631\}-\{62^2\}-\{532\}-\{541\}\\ &\quad -\{4^22\}+\{7\}+\{61\}+\{52\}+\{43\}+\{421\}-\{3\}\\ &= [12]+[10]+[9]+2[8]+[7]+3[6]+[5]+3[4]+[3]+2[2]+2[0].\end{aligned}$$

The next degree gives

$$\begin{aligned}[3]\otimes\{5\} = [15]+[13]+[12]+2[11]+2[10]+3[9]+2[8]+4[7]\\ +3[6]+4[5]+2[4]+4[3]+[2]+2[1].\end{aligned}$$

The types of the irreducible concomitants are as follows:

Degree 2; [4], [2], [0],
Degree 3; [6], [4], [3], [1],
Degree 4; [5], [3], [2], [0],
Degree 5; [4], [2], [1].

The irreducible concomitants up to degree 4 are as follows, expressing the cubic as $\alpha_x^3 = \beta_x^3 = \gamma_x^3 = \dots$.

$$\text{Degree 2; } [\alpha\beta]\,\alpha_x^2\beta_x^2, \quad [\alpha\beta]^2\,\alpha_x\beta_x, \quad [\alpha\beta]^3.$$

$$\text{Degree 3; } (\alpha\beta x)^2\,(\alpha\gamma x)\,\beta_x\gamma_x^2, \quad (\alpha\beta x)^2\,(\alpha\gamma x)\,[\beta\gamma]\,\gamma_x, \quad (\alpha\beta\gamma)^2\,\alpha_x\beta_x\gamma_x, \quad (\alpha\beta\gamma)^2\,[\alpha\beta]\,\gamma_x.$$

$$\begin{aligned}\text{Degree 4; } (\alpha\beta\gamma)^2\,(\alpha\delta x)\,\beta_x\gamma_x\delta_x^2, \quad (\alpha\beta\gamma)^2\,(\alpha\delta x)\,[\beta\gamma]\,\delta_x^2 \quad (\alpha\beta\gamma)^2\,[\alpha\delta]\,[\beta\delta]\,\gamma_x\delta_x,\\ (\alpha\beta\gamma)\,(\alpha\beta\delta)\,(\alpha\gamma\delta)\,(\beta\gamma\delta).\end{aligned}$$

## 9. The quaternary quadratic complex

The results obtained for the orthogonal quadratic in six variables may be used to find the concomitants of a quaternary quadratic complex under the full linear group.

In order to find the concomitants of a quaternary quadratic complex, Turnbull (1937) has employed the device of associating the quadratic complex with a quadratic in the 6-space whose point co-ordinates are identified with the six line co-ordinates in the 4-space. The same essential principle, though in a slightly different manner, is employed to obtain the concomitants up to degree 5 in the coefficients. The results obtained show a slight discrepancy with those of the 'complete set of irreducible concomitants' as published by Turnbull, and this drew attention to certain omissions in his list which he has since been able to correct.

If the group of transformations is restricted to be of unit determinant, the linear complex in four variables has a quadratic invariant as indicated by the following equation:

$$\{1^2\}\otimes\{2\} = \{2^2\}+\{1^4\}.$$

The relative invariant indicated by $\{1^4\}$ becomes absolute as the determinant of the transformation is made equal to unity. One may therefore replace $\{1^4\}$ by $\{0\}$ or $[0]$.

The group of transformations of the six line co-ordinates is clearly the rotation group corresponding to this quadratic invariant. It is appropriate, then, to multiply $\{1^2\}$, using the symbol $\otimes$, by a character of the orthogonal group, and accordingly

$$\{1^2\}\otimes[2] = \{1^2\}\otimes\{2\}-[0], \quad \{1^2\}\otimes[2] = \{2^2\}. \qquad (9{\cdot}1)$$

From this equation the formulae for the concomitants of a quaternary quadratic complex from the formulae for the concomitants of an orthogonal quadratic in six variables can be deduced, for

$$\{2^2\} \otimes \{n\} = \{1^2\} \otimes [2] \otimes \{n\} = \{1^2\} \otimes ([2] \otimes \{n\}).$$

When $[2] \otimes \{n\}$ has been evaluated, there remains but to find $\{1^2\} \otimes [\lambda]$ for each term $[\lambda]$ that appears. The particular $(\lambda)$ will have one, two or three parts. Since for partitions into one or two parts, $[\lambda]$ is a simple character of the rotation group, we may expect $\{1^2\} \otimes [\lambda]$ to be a single $S$-function.

Clearly $\{1^2\} \otimes [n]$ includes the principal part of $\{1^2\} \otimes \{n\}$ which is $\{n^2\}$. It may be concluded that

$$\{1^2\} \otimes [n] = \{n^2\},$$

and this is verified by counting the number of terms.

Again $$\{1^2\} \otimes [1^2] = \{1^2\} \otimes \{1^2\} = \{21^2\}.$$

Hence $\{1^2\} \otimes [n^2]$ includes the principal part of $\{21^2\}^n$ which is $\{2n, n, n\}$. Again one may conclude that

$$\{1^2\} \otimes [n^2] = \{2n, n, n\},$$

and this is verified by the counting of terms.

The principal part of $[p-q]\,[q^2]$ is $[p, q]$. Hence $\{1^2\} \otimes [p, q]$ includes the principal part of

$$(\{1^2\} \otimes [p-q])\,(\{1^2\} \otimes [q^2]) = \{p-q, p-q\}\{2q, q, q\},$$

which is $\{p+q, p, q\}$. As before, it must be assumed that

$$\{1^2\} \otimes [p, q] = \{p+q, p, q\},$$

and this is again verified by the counting of terms.

If $(\lambda)$ is a partition with three non-zero parts, then $[\lambda]$, though a simple character of the full orthogonal group, is the sum of two simple characters of the rotation group. One may expect, then, that $\{1^2\} \otimes [\lambda]$ is the sum of two $S$-functions.

Now $$\{1^2\} \otimes \{1^3\} = \{2^3\} + \{31^3\}.$$

It follows that $\{1^2\} \otimes \{n^3\}$ will certainly include the principal part of the $n$th power of this, which is $\{(2n)^3\} + \{3n, (n)^3\}$. One may conclude that

$$\{1^2\} \otimes [n^3] = \{(2n)^2\} + \{3n, (n)^3\}.$$

From this it follows, as before, that

$$\{1^2\} \otimes [p, q, r] = \{p+q+r, p, q, r\} + \{p+q, p+r, q+r\}.$$

As before, this is verified by the counting of terms.

The concomitants of a quaternary quadratic complex can now be deduced from the concomitants of an orthogonal quadratic in six variables, as obtained in § 7.

Thus
$$\begin{aligned} \{2^2\} \otimes \{2\} &= \{1^2\} \otimes [2] \otimes \{2\} \\ &= \{1^2\} \otimes ([4] + [2^2] + [2] + [0]) \\ &= \{4^2\} + \{42^2\} + \{2^2\} + \{0\}. \end{aligned}$$

It should be remembered that the restriction was made that the determinant of the transformation is unity. This had the effect of making $\{1^4\} = 1$. To remove the restriction multiply such $S$-functions in the expansion of $\{2^2\}\otimes\{n\}$ by such a power of $\{1^4\}$ as will make it a partition of $4n$.

Thus $$\{2^2\}\otimes\{2\} = \{4^2\}+\{42^2\}+\{3^21^2\}+\{2^4\}.$$

Again

$$\begin{aligned}\{2^2\}\otimes\{3\} &= \{1^2\}\otimes[2]\otimes\{3\}\\ &= \{1^2\}\otimes([6]+[42]+[4]+[31]+[2^3]+[2^2]+2[2]+[0])\\ &= \{6^2\}+\{642\}+\{5^21^2\}+\{5421\}+\{62^3\}+\{4^3\}+\{53^21\}+2\{4^22^2\}+\{3^4\}.\end{aligned}$$

Next

$$\begin{aligned}\{2^2\}\otimes\{4\} &= \{1^2\}\otimes[2]\otimes\{4\}\\ &= \{1^2\}\otimes([8]+[62]+[6]+[51]+[4^2]+[42^2]+2[42]+3[4]+[321]+[31]\\ &\quad +3[2^2]+3[2]+2[0])\\ &= \{88\}+\{862\}+\{7711\}+\{7621\}+\{844\}+\{8422\}+\{664\}+2\{7531\}+3\{6622\}\\ &\quad +\{7432\}+\{6541\}+\{6532\}+3\{6442\}+3\{5533\}+2\{4444\}.\end{aligned}$$

Lastly,

$$\begin{aligned}\{2^2\}\otimes\{5\} &= \{1^2\}\otimes[2]\otimes\{5\}\\ &= \{1^2\}\otimes([10]+[82]+[64]+[622]+[442]+[8]+[71]+2[62]+[53]+[521]\\ &\quad +[44]+[431]+[422]+3[6]+2[51]+5[42]+2[321]+2[2^3]+4[4]\\ &\quad +3[31]+3[2^2]+5[2]+2[0])\\ &= \{10.10\}+\{10.8.2\}+\{10.6.4\}+\{10.6.2.2\}+\{884\}+\{10.4.4.2\}+\{866\}\\ &\quad +\{9911\}+\{9821\}+2\{9731\}+\{9641\}+\{9632\}+\{8741\}+\{9551\}+\{9542\}\\ &\quad +\{8651\}+\{9533\}+\{7751\}+3\{8822\}+2\{8732\}+5\{8642\}+2\{8543\}+2\{7652\}\\ &\quad +2\{8444\}+2\{6662\}+4\{7733\}+3\{7643\}+3\{7553\}+5\{6644\}+2\{5555\}.\end{aligned}$$

The concept of reducibility is different from that used for the orthogonal quadratic, because the one character $[\lambda]$ of the orthogonal group will correspond to two characters of the full linear group, if the partition has three non-zero parts. The following represent the irreducible concomitants:

Degree 2; $\{422\}$, $\{3311\}$, $\{2222\}$.

Degree 3; $\{6222\}$, $\{5421\}$, $\{5331\}$, $\{444\}$, $\{4422\}$, $\{3333\}$.

Degree 4; $\{7432\}$, $\{6541\}$, $\{6532\}$, $\{6442\}$, $\{6442\}$, $\{5533\}$, $\{4444\}$.

Degree 5; $\{9542\}$, $\{8651\}$, $\{8543\}$, $\{8543\}$, $\{8444\}$, $\{7652\}$, $\{7652\}$, $\{6662\}$, $\{7643\}$, $\{7643\}$, $\{7553\}$, $\{6644\}$, $\{5555\}$.

Using the tableau $$\begin{pmatrix}\alpha & \alpha\\ \beta & \beta\end{pmatrix}$$

to denote the ground-form quadratic complex, with any number of dashes, to denote equivalent symbols, the concomitants up to degree 4 in the coefficients correspond to the following tableaux:

$$\begin{pmatrix} \alpha & \alpha & \alpha' & \alpha' \\ \beta & \beta & & \\ \beta' & \beta' & & \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha & \alpha' \\ \beta & \beta & \beta' \\ \alpha' & & \\ \beta' & & \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha \\ \beta & \beta \\ \alpha' & \alpha' \\ \beta' & \beta' \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha & \alpha' & \alpha' & \alpha'' & \alpha'' \\ \beta & \beta & & & & \\ \beta' & \beta' & & & & \\ \beta'' & \beta'' & & & & \end{pmatrix},$$

$$\begin{pmatrix} \alpha & \alpha & \alpha' & \alpha' \\ \beta & \beta & \alpha'' & \alpha'' \\ \beta' & \beta' & \beta'' & \beta'' \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha & \alpha' & \alpha' & \alpha'' \\ \beta & \beta & \alpha'' & \beta'' & \\ \beta' & \beta' & & & \\ \beta'' & & & & \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha & \alpha' & \alpha' & \alpha'' & \alpha'' & \alpha''' \\ \beta & \beta & \alpha''' & \beta''' & & & \\ \beta' & \beta' & \beta''' & & & & \\ \beta'' & \beta'' & & & & & \end{pmatrix},$$

$$\begin{pmatrix} \alpha & \alpha & \alpha' & \alpha'' & \alpha'' & \alpha''' \\ \beta & \beta & \beta' & \alpha''' & \beta''' & \\ \alpha' & \beta'' & \beta'' & \beta''' & & \\ \beta' & & & & & \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha & \alpha' & \alpha' & \alpha'' & \alpha'' \\ \beta & \beta & \alpha''' & \alpha''' & & \\ \beta' & \beta' & \beta''' & \beta''' & & \\ \beta'' & \beta'' & & & & \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha & \alpha' & \alpha' & \alpha''' & \alpha''' \\ \beta & \beta & \alpha'' & \alpha'' & & \\ \beta' & \beta' & \beta''' & \beta''' & & \\ \beta'' & \beta'' & & & & \end{pmatrix},$$

$$\begin{pmatrix} \alpha & \alpha & \alpha' & \alpha' & \alpha''' & \alpha''' \\ \beta & \beta & \alpha'' & \alpha'' & \beta''' & \\ \beta' & \beta' & \beta''' & & & \\ \beta'' & \beta''' & & & & \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha & \alpha' & \alpha'' & \alpha'' \\ \beta & \beta & \beta' & \alpha''' & \alpha''' \\ \alpha' & \beta'' & \beta'' & & \\ \beta' & \beta''' & \beta''' & & \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha & \alpha' & \alpha' \\ \beta & \beta & \alpha'' & \alpha'' \\ \beta' & \beta' & \alpha''' & \alpha''' \\ \beta'' & \beta'' & \beta''' & \beta''' \end{pmatrix}.$$

It may be noted here that the symbolic methods give a complete system without guaranteeing that all the forms retained are irreducible, and as at present found it runs up to degree 17. The methods adopted here give accurate irreducible results up to the degree found. The theories are complementary.

The above results have been checked by the counting of terms.

Now consider the concomitants of a simple quadratic complex under the orthogonal group. The ground form is then of type $[2^2]$. The concomitants of degree 2 are given by

$$\begin{aligned} [2^2] \otimes \{2\} &= (\{2^2\} - \{2\}) \otimes \{2\} \\ &= \{2^2\} \otimes \{2\} - \{2^2\}\{2\} + \{2\} \otimes \{1^2\} \\ &= \{4^2\} + \{42^2\} + \{3^2 1^2\} + \{2^4\} - \{42\} - \{321\} - \{2^3\} + \{31\} \\ &= \{4^2\} + \{4\} + \{31\} + \{2^2\} - \{42\} - 2\{2\} - \{1^2\} + 2\{0\}, \end{aligned}$$

making use of the formula (6·2) for expressing $S$-functions with three parts in terms of $S$-functions with $\leqslant 2$ parts.

Now use the formula (4·2),

$$[\lambda] = \{\lambda\} + \Sigma(-1)^{\frac{1}{2}p}\, \Gamma_{\gamma\mu\lambda}\{\mu\}.$$

With four variables the partitions $(\gamma)$ have to be summed for the terms

$$1 - \{2\} + \{31\} - \{3^2\}.$$

Hence
$$[4^2] = \{4^2\}-\{42\}+\{31\}-\{1^2\},$$
$$[4] = \{4\}-\{2\},$$
$$[2^2] = \{2^2\}-\{2\},$$
$$2[0] = 2\{0\},$$
and thus
$$[2^2]\otimes\{2\} = [4^2]+[4]+[2^2]+2[0].$$

For the degree 3,

$$\begin{aligned}[2^2]\otimes\{3\} &= \{2^2\}\otimes\{3\}-\{2^2\}\otimes\{2\}\{2\}+\{2^2\}\{2\}\otimes\{1^2\}-\{2\}\otimes\{1^3\}\\ &= \{6^2\}+\{642\}+\{62^3\}+\{5^21^2\}+\{5421\}+\{53^21\}+\{4^3\}+2\{4^22^2\}+\{3^4\}\\ &\quad -\{2\}(\{4^2\}+\{42^2\}+\{3^21^2\}+\{2^4\})+\{2^2\}\{31\}-\{41^2\}-\{3^2\}\\ &= \{6^2\}+\{642\}+\{4\}+\{4^2\}+\{431\}+\{42^2\}+\{4\}+2\{2^2\}+\{0\}-\{64\}-\{541\}-\{4^22\}\\ &\quad -\{62^2\}-\{532\}-\{52^21\}-\{4^22\}-\{4321\}-\{42^3\}-\{42\}-\{321\}-\{2^3\}-\{2\}\\ &\quad +\{53\}+\{521\}+\{431\}+\{42^2\}+\{421^2\}+\{3^22\}+\{32^21\}-\{41^2\}-\{3^2\}\\ &= \{6^2\}+\{62\}+\{53\}+\{4^2\}+3\{4\}+\{31\}+2\{2^2\}+\{0\}-\{64\}-\{6\}-\{51\}-2\{42\}\\ &\quad -3\{2\}-\{1^2\}.\end{aligned}$$

Then
$$[6^2] = \{6^2\}-\{64\}+\{53\}-\{3^2\},$$
$$[62] = \{62\}-\{6\}-\{51\}-\{42\}+\{4\}+\{31\},$$
$$[4^2] = \{4^2\}-\{42\}+\{31\}-\{1^2\},$$
$$2[4] = 2\{4\}-2\{2\},$$
$$2[2^2] = 2\{2^2\}-2\{2\},$$
$$[3^2] = \{3^2\}-\{31\}+\{2\}-\{0\},$$
$$2[0] = 2\{0\}.$$

Hence
$$[2^2]\otimes\{3\} = [6^2]+[62]+[4^2]+2[4]+2[2^2]+[3^2]+2[0].$$

In a similar manner

$$\begin{aligned}[2^2]\otimes\{4\} &= \{2^2\}\otimes\{4\}-\{2^2\}\otimes\{3\}\{2\}+\{2^2\}\otimes\{2\}\{2\}\otimes\{1^2\}-\{2^2\}\{2\}\otimes\{1^3\}+\{2\}\otimes\{1^4\}\\ &= \{8^2\}+\{84\}+\{8\}+\{75\}+\{71\}+\{6^2\}+3\{62\}+\{53\}+\{51\}+3\{4^2\}+4\{4\}\\ &\quad +4\{31\}+3\{2^2\}+3\{0\}-\{86\}-\{82\}-\{73\}-2\{64\}-3\{6\}-3\{51\}-4\{42\}-\{3^2\}\\ &\quad -6\{2\}-3\{1^2\}\\ &= [8^2]+[84]+[8]+[6^2]+2[62]+[51]+3[4^2]+3[4]+4[2^2]+3[0].\end{aligned}$$

If it is remembered that each character corresponding to partition into two parts, separates into two conjugate characters, so that the corresponding concomitant separates into two conjugate concomitants, these formulae give the types of concomitant up to degree 4 of a quadratic complex of type $[2^2]$ under the rotation group. It should be noticed that the quadratic complex itself is not simple, separating itself into two conjugate forms, but no attempt will be made here to separate the concomitants of these.

For the orthogonal group the ground form is simple, but no attempt has been made so far to discriminate between concomitants of type $[n]$ and concomitants of the associated type $[n]^*$. This could be done by following through the above working for a transformation of negative determinant, allowing a factor $-1$ every time a factor $\{1^4\}$ occurs. It is easier, however, to consider the special case for the basic transformation with matrix $A = \text{diag.}(1, 1, 1, -1)$.

This is an orthogonal matrix of negative determinant for which

$$[p, q] = 0 \quad (q \neq 0), \qquad [p] = p+1.$$

Then
$$[2^2] = 0,$$

and since the characteristic roots of $A^{[2^2]}$ are all $\pm 1$, then

$$A^{[2^2]} = \text{diag.}[1^5, (-1)^5].$$

Hence $[2^2] \otimes \{p\}$ is the coefficient of $x^p$ in the expansion of

$$(1-x^2)^{-5} = 1+5x^2+15x^4+35x^6+\ldots.$$

Thus
$$[2^2] \otimes \{2\} = 5,$$

and since the terms in $[2^2] \otimes \{2\}$ which correspond to a single part are $[4]+[0]+[0]$, and

$$[4] = 5, \quad [4]^* = -5, \quad [0] = 1, \quad [0]^* = -1,$$

it follows that
$$[2^2] \otimes \{2\} = [4^2]+[4]+[2^2]+[0]+[0]^*.$$

Again, since $[2^2] \otimes \{3\} = 0$,

$$[2^2] \otimes \{3\} = [6^2]+[62]+[4^2]+[3^2]+2[2^2]+[4]+[4]^*+[0]+[0]^*.$$

Also, because $[2^2] \otimes \{4\} = 15$, and $[8] = 9$, $[4] = 5$, $[0] = 1$,

$$[2^2] \otimes \{4\} = [8^2]+[84]+[6^2]+2[62]+[51]+[5^2]+3[4^2]+4[2^2] \\ +[8]+2[4]+[4]^*+2[0]+[0]^*.$$

The irreducible concomitants are of the following types:

Degree 2; $[4]$, $[2^2]$, $[0]$, $[0]^*$.

Degree 3; $[3^2]$, $[4]$, $[4]^*$, $[0]$, $[0]^*$.

Degree 4; $[51]$, $[4]$.

For a tensor ground form of type $[2^2]$ satisfying

$$B_{ijkp} = -B_{jikp} = -B_{ijpk} = B_{kpij},$$

e.g. the simple Riemann Christoffel tensor in its usual form, denoting $g^{ir}B_{ijkp}$ by $B^{r}{}_{.jkp}$, etc., the concomitants of degree 2 can be expressed,

$$B^{i\,.\,k}_{\,.\,j\,.\,p}B_{iqkr}, \quad B^{ij}_{\,.\,.\,kp}B_{ijqr}, \quad B^{ijkp}B_{ijkp}, \quad E^{ijqr}B_{ij}^{\,.\,.\,kp}B_{qrkp}.$$

One point of interest arises. The simple Riemann Christoffel tensor, i.e. one for which the Einstein tensor $G_{\mu\nu} = 0$, has two invariants of degree 2, one absolute and one a relative invariant. The absolute invariant $B^{ijkp}B_{ijkp}$ is well known. Eddington (1929, p. 92) refers to it as the square of the absolute value of the Riemann Christoffel tensor. Theoretically, however, in space of type $(3+1)$, e.g. space time, it could be zero or negative, though in Einstein's theory it is taken to be positive everywhere.

Apparently the relative invariant

$$E^{ijqr}B_{ij}^{\cdot\cdot kp}B_{qrkp}$$

has not previously been noticed. It is pertinent to inquire whether it has any physical significance. It is zero everywhere in Einstein's solution for a single gravitational particle, but presumably must take non-zero values in a composite field.

## 10. Concomitants under the symplectic group

To illustrate the methods as applied to the symplectic group, the complete set of irreducible concomitants of a quadratic and of a linear complex and the concomitants up to degree 3, of a quadratic complex, in the quaternary case are now dealt with.

The quadratic is the same for the symplectic group as for the full linear group, i.e.

$$\langle 2\rangle = \{2\}.$$

The concomitants of degree 2 are given by

$$\langle 2\rangle \otimes \{2\} = \{2\}\otimes\{2\} = \{4\}+\{2^2\}.$$

From (4·4) it follows that

$$\{4\} = \langle 4\rangle, \quad \{2^2\} = \langle 2^2\rangle+\langle 1^2\rangle+\langle 0\rangle.$$

Hence $\langle 2\rangle\otimes\{2\} = \langle 4\rangle+\langle 2^2\rangle+\langle 1^2\rangle+\langle 0\rangle.$

For the degree 3,

$$\begin{aligned}\langle 2\rangle\otimes\{3\} &= \{6\}+\{42\}+\{2^3\}\\ &= \{6\}+\{42\}+\{2\}\\ &= \langle 6\rangle+\langle 42\rangle+\langle 31\rangle+2\langle 2\rangle.\end{aligned}$$

For degree 4,

$$\begin{aligned}\langle 2\rangle\otimes\{4\} &= \{8\}+\{62\}+\{44\}+\{422\}+\{2^4\}\\ &= \{8\}+\{62\}+\{44\}+\{4\}+\{31\}+\{2^2\}-\{2\}-\{1^2\}+\{0\}\\ &= \langle 8\rangle+\langle 62\rangle+\langle 51\rangle+\langle 44\rangle+\langle 33\rangle+2\langle 22\rangle+\langle 11\rangle+2\langle 4\rangle+\langle 31\rangle+2\langle 0\rangle.\end{aligned}$$

For degree 5,

$$\begin{aligned}\langle 2\rangle\otimes\{5\} &= \{10\}+\{82\}+\{64\}+\{622\}+\{442\}+\{4222\}\\ &= \{10\}+\{82\}+\{64\}+\{6\}+\{51\}+\{42\}-\{4\}-\{31\}+\{42\}+\{2\}\\ &= \langle 10\rangle+\langle 82\rangle+\langle 71\rangle+\langle 64\rangle+2\langle 6\rangle+\langle 53\rangle+\langle 51\rangle+3\langle 42\rangle+2\langle 31\rangle+3\langle 2\rangle.\end{aligned}$$

The complete set of irreducible concomitants for all degrees corresponds to the following set of types:

Degree 2; $\langle 2^2\rangle$, $\langle 1^2\rangle$, $\langle 0\rangle$.

Degree 3; $\langle 2\rangle$.

Degree 4; $\langle 31\rangle$, $\langle 0\rangle$.

The calculation is much simpler than for the orthogonal group.

The irreducible concomitants can be expressed symbolically as follows.† Let $h_{ij}x^{ij}$ be the invariant linear complex and $A_{ij}x^ix^j = \alpha_x^2 = \beta_x^2 = \gamma_x^2 = \delta_x^2$ the ground-form quadratic. Denote by $(\alpha\beta\gamma h)$ the contraction

$$E^{ijkp}A_{i.}\,A_{j.}\,A_{k.}\,h_{p.},$$

by $h'_x$ the contraction $$h_{.q}x^q$$

and by $(\alpha\beta\gamma x)$ the contraction

$$(\alpha\beta\gamma h)\,h'_x = E^{ijkp}A_{i.}\,A_{j.}\,A_{k.}\,h_{pq}x^q.$$

Also $[\alpha\beta]$ denotes the contraction $h^{ij}A_{i.}\,A_{j.}$.

Then the following are the irreducible concomitants:

$$(\alpha\beta\,|\,xy)^2,\quad (\alpha\beta\,|\,xy)\,[\alpha\beta],\quad [\alpha\beta]^2.$$

$$(\alpha\beta\gamma x)^2;$$

$$(\alpha\beta\gamma x)\,(\alpha\beta\gamma h)\,(h'\delta\,|\,xy)\,\delta_x.$$

$$(\alpha\beta\gamma\delta)^2.$$

The concomitants of the linear complex under the symplectic group are easier to find than those of the quadratic. Thus

$$\langle 1^2\rangle = \{1^2\}-\{0\},$$

and
$$\langle 1^2\rangle\otimes\{2\} = \{1^2\}\otimes\{2\}-\{1^2\} = \{2^2\}+\{1^4\}-\{1^2\} = \langle 2^2\rangle+\langle 0\rangle.$$

Similarly
$$\langle 1^2\rangle\otimes\{3\} = \langle 3^2\rangle+\langle 1^2\rangle.$$

The invariant corresponding to $\langle 0\rangle$ is the only irreducible concomitant.‡

Lastly, the types of concomitant up to degree 3 of a quadratic complex are determined.

Thus
$$\langle 2^2\rangle = \{2^2\}-\{1^2\},$$

and

$$\begin{aligned}\langle 2^2\rangle\otimes\{2\} &= \{2^2\}\otimes\{2\}-\{2^2\}\{1^2\}+\{1^2\}\otimes\{1^2\}\\ &= \{4^2\}+\{422\}+\{3311\}+\{2^4\}-\{3^2\}-\{321\}-\{2^21^2\}+\{211\}\\ &= \{4^2\}+\{4\}+\{31\}+\{2^2\}-\{2\}-\{1^2\}+\{2^2\}+\{0\}-\{3^2\}-\{31\}-\{2^2\}+\{0\}-\{1^2\}\\ &\quad -\{2\}-\{1^2\}+\{0\}\\ &= \langle 4^2\rangle+\langle 4\rangle+\langle 2^2\rangle+\langle 0\rangle.\end{aligned}$$

Similarly

$$\begin{aligned}\langle 2^2\rangle\otimes\{3\} &= \{6^2\}+\{642\}+\{4^2\}+\{431\}+2\{4\}+\{42^2\}+2\{2^2\}+\{0\}-\{1^2\}(\{4^2\}\\ &\quad +\{42^2\}+\{2^2\}+\{0\})+\{211\}\{2^2\}-\{31^3\}-\{2^3\}\\ &= \{6^2\}+\{62\}+\{4^2\}+\{42\}+\{4\}+2\{2^2\}+\{0\}-\{5^2\}-\{51\}-\{3^2\}-\{31\}-2\{1^2\}\\ &= \langle 6^2\rangle+\langle 62\rangle+\langle 4^2\rangle+\langle 42\rangle+\langle 4\rangle+2\langle 2^2\rangle+\langle 0\rangle.\end{aligned}$$

† See Weitzenböck (1910) and Das Gupta (1930). The results are in agreement.

‡ See H. W. Turnbull (1926). The results tally.

The irreducible concomitants are of the following types:

$$\text{Degree } 2;\ \langle 4\rangle, \langle 2^2\rangle, \langle 0\rangle.$$
$$\text{Degree } 3;\ \langle 42\rangle, \langle 4\rangle, \langle 2^2\rangle, \langle 0\rangle.$$

## 11. Concomitants under intransitive and imprimitive groups

To illustrate the theory as applied to intransitive groups of transformations consider first the group of transformations on two sets of three variables, each transformed independently by the full linear group. The complete group is the direct product of the two full linear groups, and the characters of the complete group are products of two characters of the full linear group, or of two $S$-functions.

An $S$-function of the latent roots of the matrix of transformation on the first set of three variables is denoted by $\{\lambda\}$, and on the second set of three variables by $\{\lambda\}'$. A character of the intransitive group may thus be expressed as $\{\lambda\}\{\mu\}'$.

First determine the concomitants of a bilinear form, which is of type $\{1\}\{1\}'$.

Using the formula (Littlewood 1944*a*)

$$(AB)\otimes\{n\} = \Sigma(A\otimes\{\lambda\})\,(B\otimes\{\lambda\})$$

summed for all partitions $\{\lambda\}$ of $n$, then

$$(\{1\}\{1\}')\otimes\{2\} = \{2\}\{2\}'+\{1^2\}\{1^2\}',$$
$$(\{1\}\{1\}')\otimes\{3\} = \{3\}\{3\}'+\{21\}\{21\}'+\{1^3\}\{1^3\}',$$
$$(\{1\}\{1\}')\otimes\{4\} = \{4\}\{4\}'+\{31\}\{31\}'+\{2^2\}\{2^2\}'+\{21^2\}\{21^2\}'.\dagger$$

Clearly the irreducible concomitants consist of a bilinear contravariant of type $\{1^2\}\{1^2\}'$ and an invariant of type $\{1^3\}\{1^3\}'$.

The group of transformations may be extended to include transformations which interchange the two sets of three variables, thus forming a transitive but imprimitive group. The effect on the characters is as follows.

There will be an additional character denoted by $\{0\}^*$ which is $+1$ for the intransitive subgroup, but $-1$ for transformations which interchange the two transitive sets.

Corresponding to $\{\lambda\}\{\lambda\}'$ there is an *associated* character

$$\{\lambda\}\{\lambda\}'^* = \{0\}^*\{\lambda\}\{\lambda\}'.$$

If $\{\lambda\}\neq\{\mu\}$, the two characters $\{\lambda\}\{\mu\}'$ and $\{\mu\}\{\lambda\}'$ of the intransitive group combine to form a single character $(\{\lambda\}\{\mu\}'+\{\mu\}\{\lambda\}')$ of the imprimitive group.

The concomitants of the bilinear form are the same for the imprimitive as for the intransitive group.

It may be noticed that for this imprimitive group the linear form, of type $(\{1\}+\{1\}')$, has an irreducible bilinear concomitant of degree 2, for

$$(\{1\}+\{1\}')\otimes\{2\} = (\{2\}+\{2\}')+\{1\}\{1\}'.$$

† The result extends very easily to $n$-variables, as Professor Turnbull has pointed out to me. See H. W. Turnbull (1932).

Next find the concomitants up to degree 3 of a biquadratic form, i.e. a form of type $\{2\}\{2\}'$:

$$(\{2\}\{2\}')\otimes\{2\} = \{4\}\{4\}'+\{4\}\{2^2\}'+\{2^2\}\{4\}'+\{2^2\}\{2^2\}'+\{31\}\{31\}',$$

$$\begin{aligned}(\{2\}\{2\}')\otimes\{3\} = {} & \{6\}\{6\}'+\{6\}\{42\}'+\{6\}\{2^3\}'+\{42\}\{6\}'+\{42\}\{42\}'+\{42\}\{2^3\}'+\{2^3\}\{6\}' \\ & +\{2^3\}\{42\}'+\{2^3\}\{2^3\}'+\{51\}\{51\}'+\{51\}\{42\}'+\{51\}\{321\}'+\{42\}\{51\}' \\ & +\{42\}\{42\}'+\{42\}\{321\}'+\{321\}\{51\}'+\{321\}\{42\}'+\{321\}\{321\}' \\ & +\{411\}\{411\}'+\{411\}\{33\}'+\{33\}\{411\}'+\{33\}\{33\}'.\end{aligned}$$

The irreducible concomitants are of the following types:†

Degree 2; $\{4\}\{2^2\}'$, $\{2^2\}\{4\}'$, $\{2^2\}\{2^2\}'$, $\{31\}\{31\}'$.

Degree 3; $\{6\}\{2^3\}'$, $\{2^3\}\{6\}'$, $\{42\}\{2^3\}'$, $\{2^3\}\{42\}'$, $\{2^3\}\{2^3\}'$, $\{51\}\{42\}'$, $\{42\}\{51\}'$, $\{42\}\{42\}'$, $\{51\}\{321\}'$, $\{321\}\{51\}'$, $\{42\}\{321\}'$, $\{321\}\{42\}'$, $\{321\}\{321\}'$, $\{411\}\{411\}'$, $\{411\}\{33\}'$, $\{33\}\{411\}'$, $\{33\}\{33\}'$.

For the imprimitive group the results are similar, but each pair $\{\lambda\}\{\mu\}'$, $\{\mu\}\{\lambda\}'$ combines to form one character, and therefore represents a single concomitant.

Consider next the case of a bi-orthogonal group. It is supposed that there are two sets of three variables, each of which is transformed by independent orthogonal transformations. The characters will be the products of two characters of the ternary orthogonal group, and may be denoted by $[\lambda]\,[\mu]'$.

First consider the concomitants of a bilinear form, of which the type is $[1]\,[1]'$.

Thus

$$\begin{aligned}([1]\,[1]')\otimes\{2\} &= (\{1\}\{1\}')\otimes\{2\} = \{2\}\{2\}'+\{1^2\}\{1^2\}' \\ &= [2]\,[2]'+[2]+[2]'+[0]+[1]\,[1]',\end{aligned}$$

$$\begin{aligned}([1]\,[1]')\otimes\{3\} &= \{3\}\{3\}'+\{21\}\{21\}'+\{1^3\}\{1^3\}' \\ &= [3]\,[3]'+[3]\,[1]'+[1]\,[3]'+[1]\,[1]'+[2]\,[2]'+[2]\,[1]'+[1]\,[2]' \\ &\quad +[1]\,[1]'+[0],\end{aligned}$$

$$\begin{aligned}([1]\,[1]')\otimes\{4\} &= \{4\}\{4\}'+\{31\}\{31\}'+\{2^2\}\{2^2\}'+\{211\}\{211\}' \\ &= [4]\,[4]'+[4]\,[2]'+[2]\,[4]'+[2]\,[2]'+[4]+[4]'+[2]+[2]'+[0] \\ &\quad +[3]\,[3]'+[3]\,[2]'+[2]\,[3]'+[3]\,[1]'+[1]\,[3]'+[2]\,[2]'+[2]\,[1] \\ &\quad +[1]\,[2]'+[1]\,[1]'+[2]\,[2]'+[2]+[2]'+[0]+[1]\,[1]'.\end{aligned}$$

The irreducible concomitants are of the following types:

Degree 2; $[2]$, $[2]'$, $[1]\,[1]'$, $[0]$.

Degree 3; $[2]\,[1]'$, $[1]\,[2]'$, $[1]\,[1]'$, $[0]$.

Degree 4; $[2]\,[1]'$, $[1]\,[2]'$, $[2]$, $[2]'$, $[0]$.

To write down these concomitants explicitly it is convenient to adopt the following notation. As there is a metric in each set of variables the tensor notation may be adopted without using any contragredient suffixes, if it is assumed that every repeated suffix is contracted with the metric tensor, e.g. $A_iB_i$ signifies $g^{ij}A_iB_j$. For the special orthogonal group, i.e.

† The binary case is known. Ignoring partitions into more than two parts, the results are in agreement. See H. W. Turnbull (1923).

with metric $\Sigma(x_i)^2$, this will not differ from the ordinary summation convention. Then use is made of upper suffixes for transformations on the one set of variables, e.g. $x^1$, $x^2$, $x^3$, and lower suffixes for transformations on the other set of variables, e.g. $y_1$, $y_2$, $y_3$.

Then if $A^i_j x^i y_j$ is the bilinear form, the irreducible concomitants can be represented as follows:

$$A^i_j A^k_j x^i x^k, \quad A^j_i A^j_k y_i y_k, \quad A^i_j A^i_j, \quad E^{ijk} E_{pqr} A^i_p A^j_q x^k y_r.$$

$$E^{ikr} A^i_p A^j_p A^k_q x^j x^r y_q, \quad E_{ikr} A^p_i A^p_j A^q_k x^q y_j y_r, \quad A^i_q A^i_p A^j_p x^j y_q, \quad E^{ijk} E_{pqr} A^i_p A^j_q A^k_r.$$

$$E_{pqr} A^i_p A^j_s A^k_q A^k_s x^i x^j y_r, \quad E^{pqr} A^p_i A^s_j A^q_k A^s_k x^r y_i y_j,$$

$$E^{ijk} E^{pqr} A^i_s A^j_t A^p_s A^q_t x^k x^r, \quad E_{ijk} E_{pqr} A^s_i A^t_j A^s_p A^t_q y_k y_r, \quad A^i_p A^j_p A^i_q A^j_q.$$

As with the double linear group, the biorthogonal group can be extended to form an imprimitive group by including transformations which permute the two sets. The effect on the concomitants of a bilinear form is simply to combine pairs of non-symmetrical concomitants to form single concomitants.

Lastly the concomitants up to degree 4 of a quadratic over the imprimitive group are found. The type of a simple quadratic is $([2]+[2]')$. Now

$$([2]+[2]') \otimes \{2\} = ([4]+[4]') + ([2]+[2]') + 2[0] + [2]\,[2]',$$

$$\begin{aligned}([2]+[2]') \otimes \{3\} = {} & ([6]+[6]') + ([4]+[4]') + ([3]+[3]') + 2([2]+[2]') + 2[0] \\ & + ([4]\,[2]' + [2]\,[4]') + 2[2]\,[2]',\end{aligned}$$

$$\begin{aligned}([2]+[2]') \otimes \{4\} = {} & ([8]+[8]') + ([6]+[6]') + ([5]+[5]') + 3([4]+[4]') + 4([2]+[2]') \\ & + 3[0] + [4]\,[4]' + 3[2]\,[2]' + ([6]\,[2]' + [2]\,[6]') + 2([4]\,[2]' + [2]\,[4]') \\ & + ([3]\,[2]' + [2]\,[3]').\end{aligned}$$

The irreducible concomitants are of the following types:

Degree 2; $[2]\,[2]'$, $([2]+[2]')$, $[0]$, $[0]^*$.
Degree 3; $[2]\,[2]'$, $[2]\,[2]'^*$, $([3]+[3]')$, $[0]$, $[0]^*$.
Degree 4; $([3]\,[2]' + [2]\,[3]')$, $[2]\,[2]'$.

Using the same convention as before, these concomitants can be expressed as follows. The ground-form quadratic is taken as $A^{ij} x^i x^j + A_{ij} y_i y_j$:

$$A^{ij} A_{kp} x^i x^j y_k y_p, \quad A^{ij} A^{ik} x^j x^k + A_{ij} A_{ik} y_j y_k, \quad A^{ij} A^{ij} + A_{ij} A_{ij}, \quad A^{ij} A^{ij} - A_{ij} A_{ij};$$

$$A^{ij} A^{ik} A_{pq} x^j x^k y_p y_q + A_{ip} A_{iq} A^{jk} x^j x^k y_p y_q, \quad A^{ij} A^{ik} A_{pq} x^j x^k y_p y_q - A_{ip} A_{iq} A^{jk} x^j x^k y_p y_q,$$

$$E^{pqr} A^{pi} A^{qj} A^{ik} x^r x^j x^k + E_{pqr} A_{pi} A_{qj} A_{ik} y_r y_j y_k, \quad A^{ij} A^{jk} A^{ki} + A_{ij} A_{jk} A_{ki}, \quad A^{ij} A^{jk} A^{ki} - A_{ij} A_{jk} A_{ki};$$

$$E^{pqr} A^{pi} A^{qj} A^{ik} A_{st} x^r x^j x^k y_s y_t + E_{pqr} A_{pi} A_{qj} A_{ik} A^{st} x^s x^t y_r y_j y_k, \quad A^{ij} A^{ik} A_{pq} A_{pr} x^j x^k y_q y_r.$$

As regards reducibility, the convention has been adopted that in the product of

$$([p]\,[q]' + [q]\,[p]') \quad \text{and} \quad ([r]\,[s]' + [s]\,[r]'),$$

if $p>q$ and $r>s$, the principal part is $([p+r]\,[q+s]' + [q+s]\,[p+r]')$.

Except for the failure of the unmodified method to deal with the case of semitransitive groups (step transformations), these methods involving group characters and the use of the symbol $\otimes$, would thus seem to have universal applicability to the problems of invariant theory.

It is hoped to discuss in another paper some modifications which will make the theory applicable also to semitransitive groups.

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